Math 2270-3
Tuesday Dec. 1.

Do the last example illustrating the geometric meaning of complex scalar multiplication, page 4 Wed Nov. 25 notes

- Then work through the complex vector space math in Monday's notes, using the glucose-insulin model to explore the issues which arise.

In G-H model, $A_{2\times2}$ has 2 distinct complex eigenvalues, $\lambda = a \pm bi$ $b \neq 0$, so is diagonalizable in $\mathbb{C}^2$ (because all our theorems about diagonalizability still hold if we switch to complex vector spaces).

In Monday notes we used the complex eigenbasis $\left\{ \begin{bmatrix} -2 \\ i \end{bmatrix}, \begin{bmatrix} 2 \\ i \end{bmatrix} \right\}$ for $\mathbb{C}^2$.

There is an alternate approach to understand discrete dynamical systems for transition matrix $A_{2\times2}$ with complex eigenvalues, based on rotation-dilation matrices:

Let $A_{2\times2}$ real entries, $A\vec{w} = \lambda\vec{w}$ with $\lambda = a+bi$, $\vec{w} = \vec{u} + i\vec{v}$

so

$$A(\vec{u}+i\vec{v}) = A\vec{u} + iA\vec{v}$$
$$= (a+ib)(\vec{u}+i\vec{v})$$
$$= (a\vec{u}-b\vec{v}) + i(b\vec{u}+a\vec{v})$$

so $A\vec{u} = a\vec{u} - b\vec{v}$
$A\vec{v} = b\vec{u} + a\vec{v}$

so, for $\mathcal{B} = \{\vec{v}, \vec{u}\}$, $T(\vec{x}) := A\vec{x}$,

$$B = [T]_{\mathcal{B}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$\uparrow [T(\vec{v})]_{\mathcal{B}}$ $\quad$ $\uparrow [T(\vec{u})]_{\mathcal{B}}$

$$= r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

dilation $r = \sqrt{a^2+b^2}$. $\quad$ rotation

looking at right.

$$B = S^{-1}AS$$

$$[\vec{v}|\vec{u}] = S_{E\leftarrow\mathcal{B}}$$

so $A = SBS^{-1}$

so soln $\vec{x}(t)$ to $\begin{cases} \vec{x}(t+1) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$ is given by

(over!)

$$\vec{x}(t) = A^t \vec{x}_0$$

$$= \underbrace{S\, B^t \underbrace{S^{-1}\vec{x}_0}_{\text{coords of } \vec{x}_0 \text{ in } \mathcal{B}\text{-basis}}}$$

$$B^t = r^t \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^t = r^t \underbrace{\begin{bmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{bmatrix}}_{\text{rotate by } \theta t} \qquad t = 0, 1, 2, \dots$$

rotate by θt, also interpolates discrete points with a smooth curve for $t \in \mathbb{R}$.

So $B^t S^{-1}\vec{x}_0$ is a sequence of points on a circular spiral, spiraling $\to 0$ if $0 < r < 1$
$\to \infty$ if $r > 1$
on unit circle if $r = 1$

So $S\left[B^t S^{-1}\vec{x}_0\right]$ is an elliptical spiral.

<u>Example</u>: $\begin{bmatrix} G(t+1) \\ H(t+1) \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$

$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$.

from previous work,

$$A \begin{bmatrix} -2 \\ i \end{bmatrix} = (\overset{a}{.9} + \overset{b}{.2}i) \begin{bmatrix} -2 \\ i \end{bmatrix}$$

$$A(\underset{\vec{u}}{-2\vec{e}_1} + i\underset{\vec{v}}{\vec{e}_2}) =$$

$\mathcal{B} = \{\vec{e}_2, -2\vec{e}_1\}$, $B = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\qquad \theta = \arctan 2/9$

$S = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} G(t) \\ H(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} (\sqrt{.85})^t \underbrace{\begin{bmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 0 \\ -50 \end{bmatrix}}}_{\begin{bmatrix} 50 \sin\theta t \\ -50\cos\theta t \end{bmatrix}}$$

$$= (.85)^{t/2} \begin{bmatrix} 100 \cos\theta t \\ 50 \sin\theta t \end{bmatrix} \quad \checkmark$$

§7.5 Stability

For many discrete dynamical systems with transition matrix $A_{n\times n}$ it is important to know whether the zero solution $\vec{x}(t)=\vec{0}$ $\forall\, t=0,1,2,\ldots$ is stable ("asymptotically stable", to be precise.

That is, is
$$\lim_{t\to\infty} \vec{x}(t) = \vec{0} \qquad \text{for all choices of initial vector } \vec{x}_0.$$
(think of G-H model.)

Theorem the zero solution of a discrete dynamical system is asymptotically stable iff all eigenvalues of A have modulus <1.

proof is easy if A is diagonalizable, harder if A is not diagonalizable (related to 2280 ideas)

let $\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_n\}$ be a complex eigenbasis for A

write $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

then $\vec{x}(t) = A^t\vec{x}_0 = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n$

$$|\lambda_i^t| = |\lambda_i|^t \to 0 \ \forall i \quad \text{iff} \quad |\lambda_i|<0 \ \forall i$$

∎

(now you can do §7.5 hw.)

<u>Wed</u>: the discrete dynamical system behind google page rank.
coincidentally, that also the topic of Wed undergraduate colloquium!

12:55
Prof. Maritza Sirvent
LCB 225
Pizza too!