Math 2270-3
Monday 31 Aug.

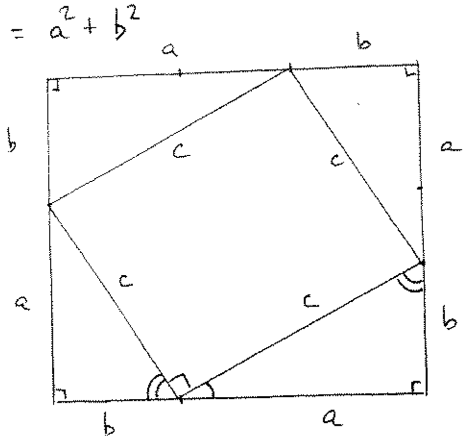
Appendix A

$\underbrace{\text{Appendix A}}$ linear geometry review for $\mathbb{R}^2$ & $\mathbb{R}^3$

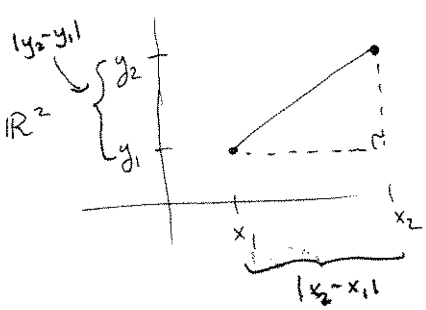
we've already defined scalar multiplaction of vectors, and vector addition, and we understand what these operations mean geometrically, in terms of displacements.
Now we want to talk about lengths (and distances) in Euclidean space:

Use this tile diagram, equating two computations of the total area, to deduce Pythagorean Thm for right Δ's (legs a, b, hypotenuse c), i.e.

$$c^2 = a^2 + b^2$$

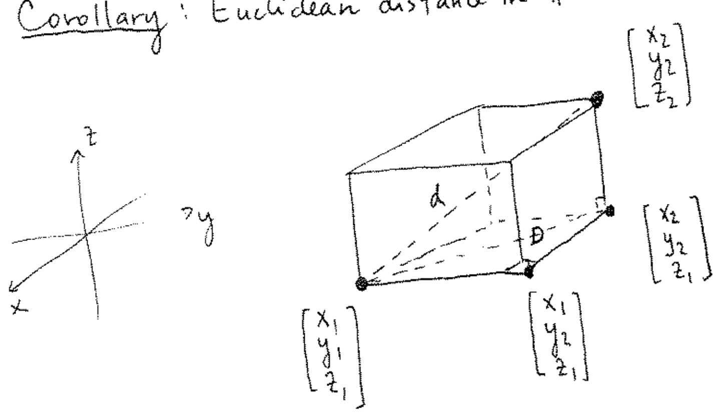


Deduce
Corollary: Euclidean distance in $\mathbb{R}^2$



$$d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

Corollary: Euclidean distance in $\mathbb{R}^3$



$$d^2 = D^2 + (z_2-z_1)^2$$
$$= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Dot product, angles, and orthogonality

If $\vec{u}, \vec{v}$ are vectors with n components, define the dot product $\vec{u}\cdot\vec{v} := \sum_{i=1}^{n} u_i v_i$

Def: e.g. $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v} = [3, 2, 1]$, then $\vec{u}\cdot\vec{v} = 3 - 2 + 2 = 3$

for the dot product, we don't care if vectors are row vectors of column vectors. (For matrix multiplication we will!

Definition: the magnitude or length of a vector $\vec{u}$ is defined to be

$$\|\vec{u}\| := \left(\sum_{i=1}^{n} u_i^2\right)^{1/2} = \sqrt{\vec{u}\cdot\vec{u}}$$

We showed that in $\mathbb{R}^2$ & $\mathbb{R}^3$, the magnitude is the distance between the initial and terminal points of the displacement $\vec{u}$. This will also be true in $\mathbb{R}^n$, for any $n = 2, 3, 4 \ldots$

Algebra of the dot product: Check these!

A.5 p.442:

1. $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$ dot product is commutative

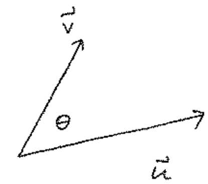
2. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$ distributive property of dot product over addition

3. If $k \in \mathbb{R}$, $(k\vec{v})\cdot\vec{w} = k(\vec{v}\cdot\vec{w}) = \vec{v}\cdot k\vec{w}$

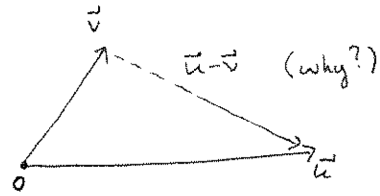
4. $\vec{v}\cdot\vec{v} > 0$ for every ~~nonz~~ $\vec{v} \neq \vec{0}$ (and $\vec{0}\cdot\vec{0} = 0$).

<u>Theorem</u>: In $\mathbb{R}^2$ & $\mathbb{R}^3$ (and later $\mathbb{R}^n$)

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$, where $\measuredangle \vec{u},\vec{v} = \theta$:



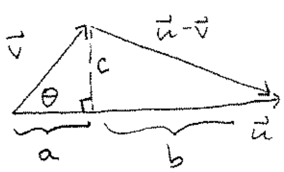
<u>proof</u>: (In case θ is an acute angle... other case analogous)
Assume $\|\vec{v}\| \le \|\vec{u}\|$ too; else relabel



compute $\|\vec{u}-\vec{v}\|^2$ two ways: once algebraically and once geometrically

① $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$; expand using algebraic properties!

② 2 Pythag. applications



$\|\vec{u}-\vec{v}\|^2 = b^2 + c^2$

... finish & deduce result!
(use geometry to deduce a, c; hence b.)

(Equating your expressions for $\|\vec{u}-\vec{v}\|^2$ from ①,② should yield the result!)

<u>Example</u>: Find the angle between $[1,-1,3]$ and $[2,2,1]$!!

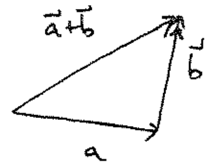
<u>Cor</u> $\vec{u}\cdot\vec{v} = 0$ iff $\theta = \pi/2$, i.e. $\vec{u} \perp \vec{v}$. This is true in $\mathbb{R}^2, \mathbb{R}^3$. In $\mathbb{R}^n$ this is how we <u>define</u> perpendicular vectors.

$\uparrow$ is perpendicular to (orthogonal)

<u>Cor</u> $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$ (At least in $\mathbb{R}^2$ & $\mathbb{R}^3$ ~ discuss $\mathbb{R}^n$ truth in chapter 5).

<u>Easy Pythagorean Theorem in $\mathbb{R}^n$</u>. Using the definition that $\vec{a} \perp \vec{b}$ iff $\vec{a}\cdot\vec{b} = 0$ (which agrees with $\theta = \pi/2$ in $\mathbb{R}^2, \mathbb{R}^3$)

Then $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$



<u>proof</u>:

<u>Example</u> Are $\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\0\end{bmatrix}$ orthogonal?

<u>Example</u> Find all vectors in $\mathbb{R}^3$ perpendicular to $\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$? (use a linear system)

(you could also use the cross product, if you know about it.)

For more review see appendix A. Tomorrow begin Chapter 2.