Math 2270-3
Friday Aug. 28

Homework for Friday Sept. 3
§1.2 17, 18, 20, 21, 24, 25, 29, 30, 32, 34, 35, 37, 38, 41
§1.3 1, 5, 6, 7, 9, 11, 13, 14, 17, 18, 20, 22, 24, 27, 34, 35, 55
§2.1 1, 3, 4, 5, 6, 7, 8, 10, 13, 16, 17, 19, 24, 25, 26

§1.3 cont'd.

- finish the 2 examples on page 4 of Wednesday's notes.

Then let's think about the big picture:

Consider the linear system

$$(LS)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\underset{\substack{\uparrow \\ \#\text{rows}}}{A}{}_{m\times n} \overset{}{\underset{\#\text{cols}}{}} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \qquad \vec{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We solve this system by computing the reduced row echelon form of A augmented by b
(the "augmented" matrix)

$$\operatorname{rref}\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & & \vdots & \vdots \\ \vdots & & & \\ a_{m1} & & a_{mn} & b_m \end{array}\right] = \operatorname{rref}(A \mid \vec{b})$$

Let's think about the universe of possibilities for the solution set, in terms of

\# rows (A) (m)
\# cols (A) (n)
\# non-zero rows in rref(A) $\longrightarrow$ called the <u>rank</u> of A, rank(A)
(maybe also consider rank(A|b) in this discussion.)

Example :

$[A \mid \vec{b}]$

$$\left[\begin{array}{ccccc|c} 3 & 6 & 7 & 2 & 5 & 10 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right]$$

$\text{rref}\,[A \mid b]$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- What is the general solution?
  (Call the unknowns $x_1, x_2, \dots x_5$)

- How many "free variables" appear in the general solution?
  (i.e. free parameters)

<u>General Case Questions</u>:

- If the system has <u>no</u> sol'ns it's called inconsistent.
  How is this reflected in $\text{rref}(A \mid b)$?

(over)

- What must rref$(A \vdots b)$ look like in order that the system have a unique (single) solution? Express this in terms of m, n and rank(A), rank$(A \vdots b)$
  Give examples

- Under exactly what conditions will the system have ∞'ly many sol's? (in terms of m, n, rank(A), rank$(A|b)$)

- If the system is consistent, express the number of free variables in terms of m, n, rank(A). Give examples.

Note: to test your reasoning ability further, check out T-F questions, page 38-40

Linear combination interpretation of LS

You've probably been taught how to multiply a matrix times a vector:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \quad \left(= \begin{bmatrix} Row_1(A)\cdot\vec{x} \\ Row_2(A)\cdot\vec{x} \\ \vdots \\ \underset{\uparrow \text{ dot product}}{Row_m(A)\cdot\vec{x}} \end{bmatrix} \right)$$

So we can write the page 1 (LS) much more compactly as

$$(LS) \qquad A\vec{x} = \vec{b}$$

There's another, equally important way to write (LS), called the <u>linear combination form</u>.

A <u>linear combination</u> of a collection of vectors is a sum of scalar multiples of those vectors.

Notice that

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_n \, col_n(A).$$

So we may also write, and interpret (LS) as

$$(LS) \qquad x_1 \, col_1(A) + x_2 \, col_2(A) + \cdots + x_n \, col_n(A) = \vec{b}$$

i.e. we're trying to solve a linear combination problem, not necessarily an intersecting hyperplane one!

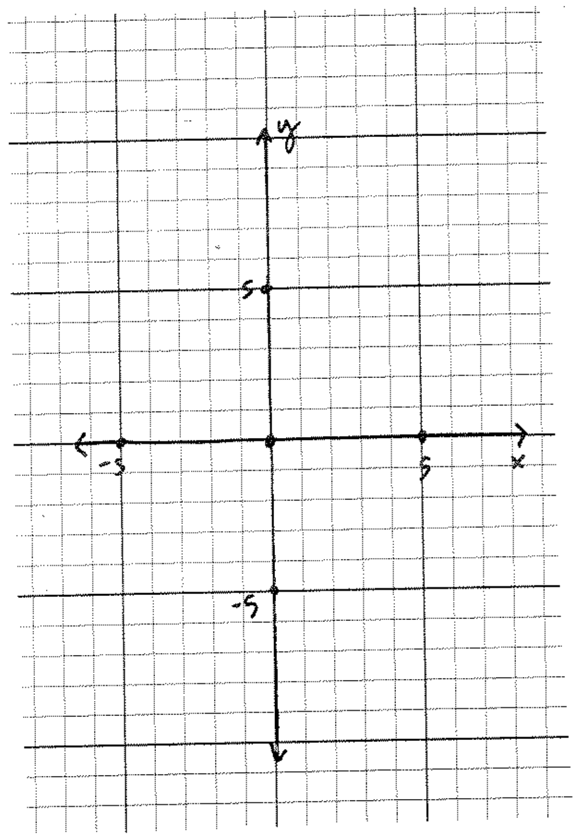
See example!

<u>Example</u> Interpret $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ as an intersecting line problem (call $x_1 = x$, $x_2 = y$)

<u>and</u> as a linear combo problem (call $x_1 = r$, $x_2 = t$)

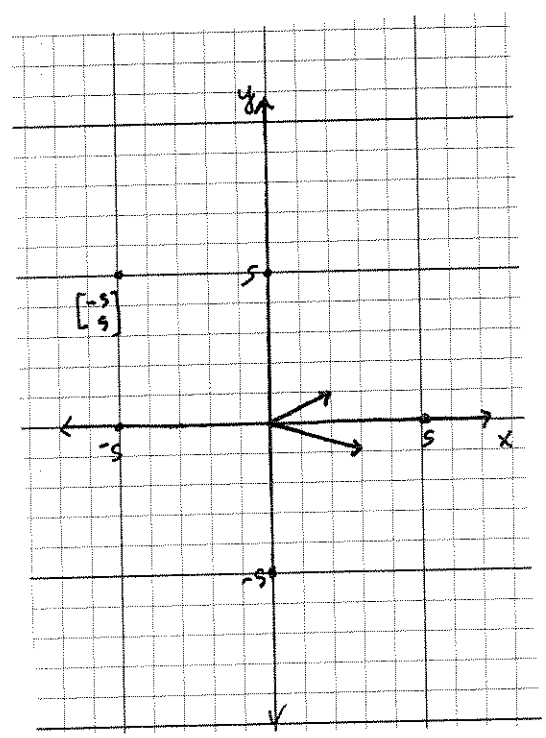
intersecting lines:

$$\begin{aligned} 2x + 3y &= -5 \\ x - y &= 5 \end{aligned}$$



linear combination problem:

$$r\begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$



alg ans: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

On Monday we'll go over appendix A: vectors & the dot product in Euclidean geometry.