Math 2270-3
Wed. Aug 26
§1.2-1.3

on HW for Friday, §1.2 #16,17 are now not required, since our computer lab doesn't open until Monday.

Do you remember how we write the general linear system of m equations in n unknown variables $x_1, x_2, \ldots x_n$?

At the end of yesterday's class we had simplified the system

$$\begin{aligned}
3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 &= 53\\
7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 &= 105\\
-4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 &= 11
\end{aligned}$$

by first writing the coefficient matrix, augmenting it with the right hand side values:

$$\left[\begin{array}{ccccc|c}
3 & 6 & 9 & 5 & 25 & 53\\
7 & 14 & 21 & 9 & 53 & 105\\
-4 & -8 & -12 & 5 & -10 & 11
\end{array}\right]$$

every matrix has one and only one reduced row echelon form, no matter what elementary row ops you use to find it — proof in chapter 3!

and then using elementary row operations to get the reduced row echelon form for the augmented matrix

$$\left[\begin{array}{ccccc|c}
1 & 2 & 3 & 0 & 5 & 6\\
0 & 0 & 0 & 1 & 2 & 7\\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

a) in every non-zero row, the first non-zero entry is a 1, called a "leading 1"

b) every column containing a (some row's) leading 1, has a zero in every other column entry

c) the leading 1's in successive rows move strictly to the right (by at least 1 column).

so the original system is equivalent to the simpler one:

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 \qquad + 5x_5 &= 6\\
x_5 + 2x_6 &= 7
\end{aligned}$$

Remember how we used Gaussian elimination to get the reduced row echelon form. Bretscher (page 15) does it slightly differently. Do you see the difference?

Let us summarize.

Solving a system of linear equations

We proceed from equation to equation, from top to bottom.

Suppose we get to the ith equation. Let x_j be the leading variable of the system consisting of the ith and all the subsequent equations. (If no variables are left in this system, then the process comes to an end.)

- If x_j does not appear in the ith equation, swap the ith equation with the first equation below that does contain x_j.
- Suppose the coefficient of x_j in the ith equation is c; thus this equation is of the form $cx_j + \cdots = \cdots$. Divide the ith equation by c.
- Eliminate x_j from all the other equations, above and below the ith, by subtracting suitable multiples of the ith equation from the others.

Now proceed to the next equation.

If an equation *zero* = *nonzero* emerges in this process, then the system fails to have solutions; the system is *inconsistent*.

When you are through without encountering an inconsistency, solve each equation for its leading variable. You may choose the nonleading variables freely; the leading variables are then determined by these choices.

After reducing the system, we "backsolved":

$x_5 = t$ (any #)
$x_4 = 7-2t$
$x_3 = s$
$x_2 = r$
$x_1 = 6-5t-3s-2r$

, so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-5t-3s-2r \\ r \\ s \\ 7-2t \\ t \end{bmatrix}$

gives all possible solns in terms of 3 free parameters.

(There would be other ways to do this, for example in this problem we could've chosen x_4 as a free parameter instead of x_5.)

To understand what these free parameters mean geometrically we need to review vector algebra & geometry...

Vector algebra and geometry

a vector $\vec{v} \in \mathbb{R}^n$ is an ordered n-tuple $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ of real numbers (scalars).

($\in$: an element of)

for $t \in \mathbb{R}$, $\quad t\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \overset{\text{is defined as}}{:=} \begin{bmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_n \end{bmatrix}$ scalar multiplication

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} := \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix}$$ vector addition

n-vectors can represent displacements in $\mathbb{R}^n$

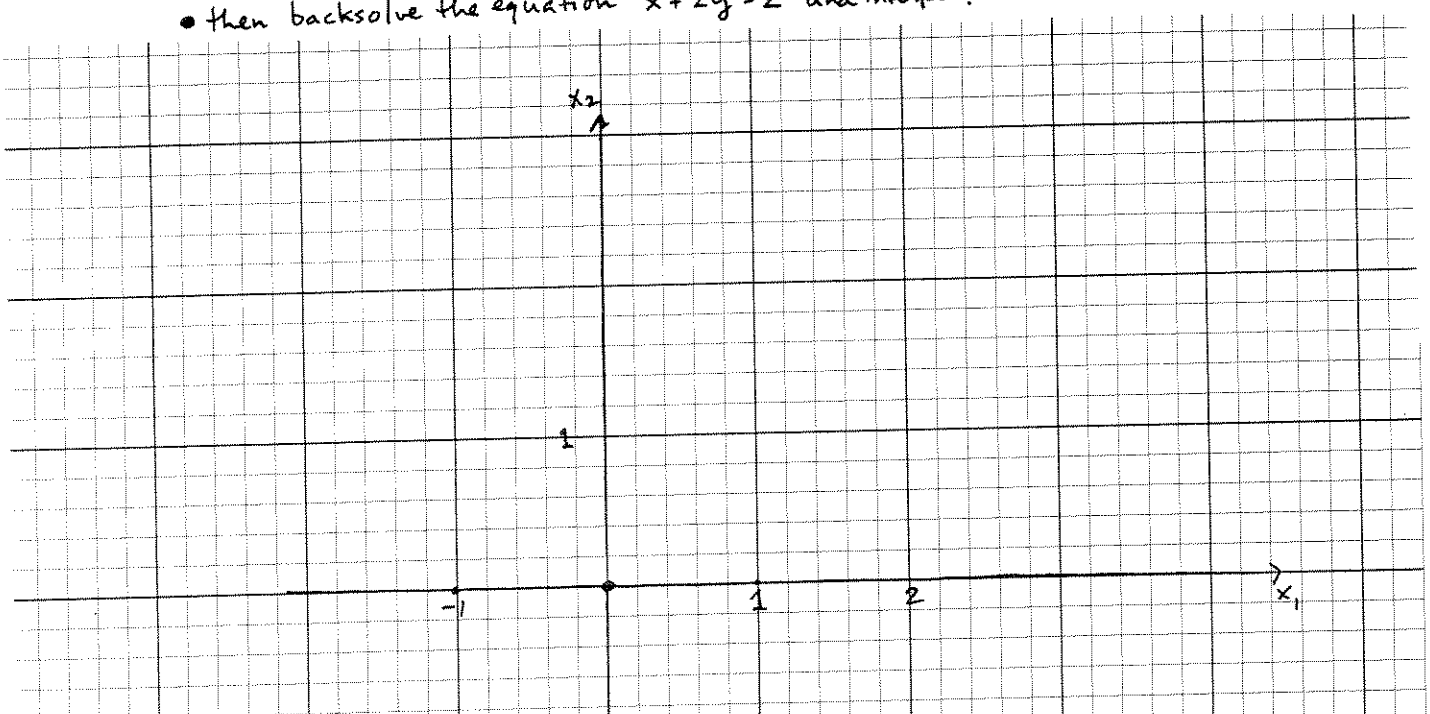
(which is one reason we use the arrow notation $\vec{v}$)

scalar multiplication of a vector yields (or represents) a parallel displacement, in the same direction if scalar > 0
opposite direction if scalar < 0

vector addition corresponds to net displacement, i.e.
$\vec{v} + \vec{w}$ is net dispacement of successively displacing by $\vec{v}$, then $\vec{w}$ (or vice verse, $\vec{v}+\vec{w} = \vec{w}+\vec{v}$).

Example: • Represent $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $-\frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as displacements from the origin (also called position vectors)

• then backsolve the equation $x + 2y = 2$ and interpet!



Example: Rewrite your solution on page 2 in linear combination form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6-5t-3s-2r \\ r \\ s \\ 7-2t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}}_{\text{const vector}} + r\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} + s\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} + t\begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}.$$

Example Solve the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 - x_2 &= 1 \\ -5x_2 + 2x_3 &= 1 \end{aligned}$$

by computing rref of augmented matrix and backsolving.

Interpret geometrically.

Example What about the almost identical system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 - x_2 &= 1 \\ -5x_2 + 2x_3 &= \underline{\underline{5}} \end{aligned} \qquad ?$$

What happened?