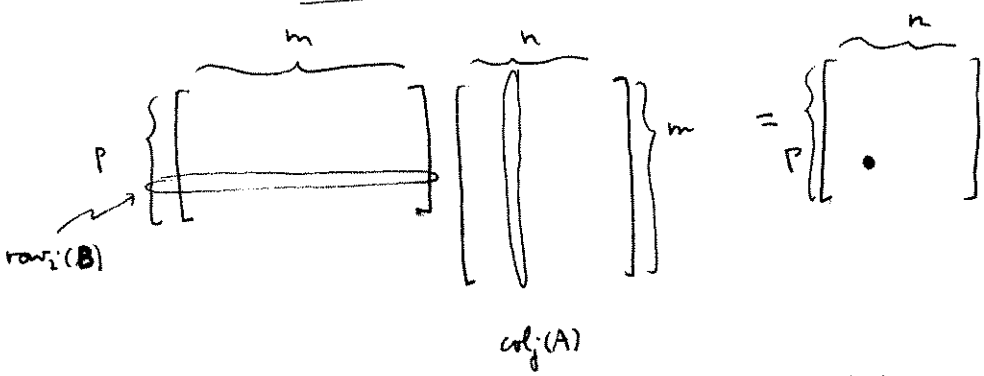
Math 2270-1
Wed 7 Sept.
§2.2–2.4

[Matrix products and composition of matrix transformations
Inverse matrices and inverses of matrix transformations

Def : Let $B_{p\times m}$, $A_{m\times n}$.

Then $(BA)_{p\times n}$ is defined by

$$\boxed{\text{entry}_{ij}(BA) := \text{row}_i(A)\cdot\text{col}_j(B)}$$



notice, this definition is equivalent to

$$\boxed{\text{col}_j(BA) = B\,\text{col}_j(A)}$$

Example

Compute $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix} =$

Now rediscuss page 3 Tuesday!

<u>Example</u> $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$

$$g\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Find $g(f(\vec{x}))$; compare to page 1 matrix product!

<u>Example</u> Do the composition of rotations example on page 3 Tuesday.

matrix for rot. by α: $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

matrix for rot by β: $\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

rot by $\alpha+\beta$:

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Example : What is the geometric effect of 1^{st} reflecting across the line $y=x$, and then across the y-axis? (page 2 Tues.)

Inverse transformations and inverse matrix

For $f(\vec{x}) = A_{m\times n}\vec{x}$, $f: \mathbb{R}^n \to \mathbb{R}^m$, when does there exist an inverse function $g: \mathbb{R}^m \to \mathbb{R}^n$, i.e. so that $g(f(\vec{x})) = \vec{x} \ \forall \vec{x}$, $f(g(\vec{y})) = \vec{y} \ \forall \vec{y}$?

Answer : f must be 1 to 1 : for each $\vec{y}$ in the range of f there must be exactly one $\vec{x} \in \mathbb{R}^n$ so that $f(\vec{x}) = \vec{y}$

& f must be onto : Every $\vec{y} \in \mathbb{R}^m$ must be $f(\vec{x})$ for some $\vec{x} \in \mathbb{R}^n$

These are the two conditions (in general) which are necessary and sufficient for an inverse function to exist.

$n > m$: f is not 1-1 (more cols than rows). Consider sol's to $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccccc|c} 1 & \cancel{*} & \cdots & & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots \\ 0 & 0 & 0 & 1 \cdots & 0 & \vdots \\ & & & & & 0 \end{array}\right]$$

$n > m \Rightarrow \exists$ col. without leading 1 in rref $\Rightarrow \infty$'ly many sol's to $A\vec{x} = \vec{0}$ $\therefore$ f not 1-1

$n < m$ (more rows than cols) f is not onto!

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & ? \\ a_{21} & \cdots & a_{2n} & ? \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} & ? \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & ? \\ 0 & 1 & \cdots & 0 & \vdots \\ & & \vdots & & \\ 0 & 0 & 0 & 0 & ? \end{array}\right]$$

$\to$ at least one zero row in rref(A). If augment with ? vector having a "1" in last entry & reverse row ops, work backwards to an inconsistent system $A\vec{x} = \vec{b}$. i.e. $\vec{b}$ is not in range(f)!

$n = m$: A is a square matrix. : Solve $A\vec{x} = \vec{y}$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & & & & y_2 \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & & a_{nn} & y_n \end{array}\right]$$

$\operatorname{rref}(A) = I$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & & 0 & z_1 \\ 0 & 1 & \cdots & & 0 & z_2 \\ 0 & 0 & 1 & \cdots & 0 & z_3 \\ & & \vdots & & & \\ 0 & 0 & \cdots & & 0\ 1 & z_n \end{array}\right]$$

notice there is a unique solution $\vec{x} = \vec{z}$ in this case, and the formulas for each ~~component~~ z_i are linear combos of the $y_1, \ldots, y_n$, so

$\vec{z} = B\vec{y}$ for some matrix B!

Inverse exists and is a matrix transformation!

$\operatorname{rref}(A) \neq I$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & \cdots & & 0 & z_1 \\ 0 & 1 & \cdots & & 0 & z_2 \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & z_n \end{array}\right]$$

If $\operatorname{rref}(A)$ has at least bottom row of zeroes, and at least one col without a leading 1. So f is neither 1-1 nor onto.

No inverse exists!

We write A^{-1} for B.

Since

$g(f(\vec{x})) = \vec{x}$, $B(A(\vec{x})) = (BA)\vec{x} = I\vec{x} \quad \forall \vec{x}$

$f(g(\vec{y})) = \vec{y}$, $A(B(\vec{y})) = (AB)\vec{y} = I\vec{y} \quad \forall \vec{y}$

so $BA = I$

$AB = I$

How to find A^{-1}!

$$A\left[\,col_1(B) \;\vdots\; col_2(B) \;\vdots\; \cdots\; col_n(B)\,\right] = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & \vdots & 1 & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

find all n col's at once synthetically!

$$\left[\; A \;\begin{array}{|cccc} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \cdots & \vdots \\ 0 & 0 & & 1 \end{array}\right]$$

$$\Big\downarrow \text{rref}$$

$$\left[\; I \;\Big|\; \underbrace{col_1(B) \;\Big|\; col_2(B) \;\Big|\; \cdots col_n(B)}_{B\,!} \right]$$

<u>example</u>: Find $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}^{-1}$, and check!

$$\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ \hline & & & \end{array}$$