Math 2270
Tues 6 Sept.

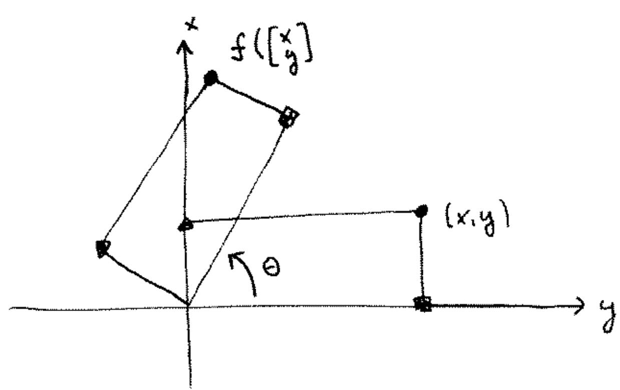
Transform Bob by affine transformations! - page 3 Fri.

$$a\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ax + cy + e \\ bx + dy + f \end{bmatrix} = x\begin{bmatrix} a \\ b \end{bmatrix} + y\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

Special matrix transformations:

- rotate by θ radians counterclockwise:

Let's work out the formula, using trig:



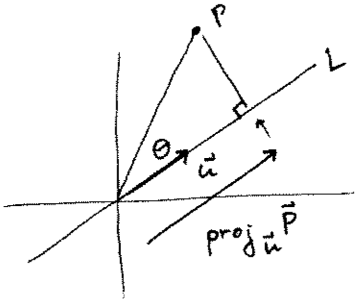
ans:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

use this, plus a translation, to write down the formula for the rotated & translated Bob on page 3 Fri

Reflect through (across) <u>any</u> line L passing through the origin:

1^{st}, recall <u>projections</u> from 2210 or 1260:



$\vec{u}$ a <u>unit</u> vector in dir. of L (i.e. $\|\vec{u}\| = 1$)

$$\text{proj}_{\vec{u}} P := \|P\| \cos\theta \, \vec{u}$$
$$= (\vec{u} \cdot \vec{P}) \vec{u} \qquad \text{since } \vec{u} \cdot \vec{P} = \|\vec{u}\| \, \|\vec{P}\| \cos\theta = \|\vec{P}\| \cos\theta$$

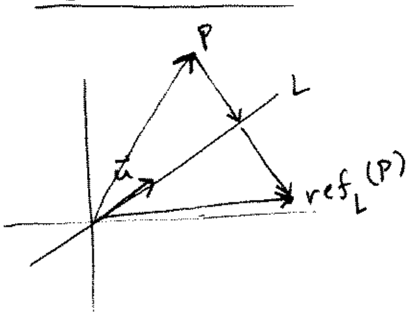
<u>example</u> If L is the line $y = x$

$$\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\text{proj}_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x+y) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

general formula, for $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$?

<u>reflection thru L</u>:



$$\text{ref}_L(\vec{P}) := \vec{P} + 2\left[\text{proj}_L \vec{P} - \vec{P}\right]$$
$$= -\vec{P} + 2 \, \text{proj}_L \vec{P}$$
$$= -\vec{P} + 2 (\vec{u} \cdot \vec{P}) \vec{u}$$
$$\text{ref}_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -\begin{bmatrix} x \\ y \end{bmatrix} + 2(u_1 x + u_2 y) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 + 2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & -1 + 2u_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} u_1^2 - u_2^2 & 2u_1u_2 \\ 2u_1u_2 & u_2^2 - u_1^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{since } u_1^2 + u_2^2 = 1$$

<u>example</u>: Check that this formula works for reflection thru the line $y = x$

(ans: $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$)

Compositions of matrix transformations

$$f(\vec{x}) = A_{m\times n}\vec{x} = x_1 \,col_1(A) + x_2\, col_2(A) + \cdots + x_n\, col_n(A) = \begin{bmatrix} row_1(A)\cdot\vec{x} \\ row_2(A)\cdot\vec{x} \\ \vdots \\ row_m(A)\cdot\vec{x}\end{bmatrix}$$

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$
$$A(k\vec{x}) = kA\vec{x}$$

checked Friday!

Suppose
$$g(\vec{y}) = B_{p\times m}\vec{y}$$

Then $g(f(\vec{x}))$ makes sense! $\quad \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$

What is it?

$$\begin{aligned} g(f(\vec{x})) &= g(A\vec{x}) \\ &= B\left(x_1 col_1(A) + x_2 col_2(A) + \cdots + x_n col_n(A)\right) \\ &= x_1 B col_1(A) + x_2 B col_2(A) + \cdots + x_n B col_n(A) \\ &= C\vec{x} \qquad \text{where } C = \left[\, B col_1(A) \,\vdots\, B col_2(A) \,\vdots\, \cdots \, B col_n(A)\right] \end{aligned}$$

$$:= BA.$$

this is how we multiply matrices!

<u>Example</u>: Use the fact that if you compose rotation by α with rotation by β, you get rotation by $\alpha+\beta$, to deduce the angle addition formulas for cos & sin!