Math 2270-1
Wed 28 Sept

§3.4 Coordinates and linear tranformation matrices, with respect to arbitrary bases

Recall (26 Sept)

If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ basis for W, call the basis $\mathcal{B}$

and $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$

then the linear combo coeff's of $\vec{w}$ are called the coords of $\vec{w}$ with respect to $\mathcal{B}$, $[\vec{w}]_{\mathcal{B}}$

$$\vec{w} = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}}\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

i.e. $\boxed{\vec{w} = S\,[\vec{w}]_{\mathcal{B}}}$, and if $W = \mathbb{R}^n$ so $S_{n\times n}$, then also $S^{-1}\vec{w} = [\vec{w}]_{\mathcal{B}}$

New today:

Matrix of a linear transformation $f(\vec{x}) = A\vec{x}$, $f: \mathbb{R}^n \to \mathbb{R}^n$ <u>with respect to a basis</u> $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$.

[This will not be A in general, unless $\mathcal{B} = E$, the standard basis].

This matrix B is to have the property that

$$\boxed{B\,[\vec{x}]_{\mathcal{B}} = [f(\vec{x})]_{\mathcal{B}}}$$

i.e. B tells how to transform the $\mathcal{B}$-coords of $\vec{x}$ to get the $\mathcal{B}$-coords of $f(\vec{x})$.

The following diagram shows that B exists, and how to compute it using A and S:

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, dashed, shift right, "S^{-1}"'] & f(\vec{x}) \arrow[d, shift left, "S^{-1}"] \\ {[\vec{x}]_\beta} \arrow[r, "B"] \arrow[u, shift right, "S"'] & {[f(\vec{x})]_\beta} \arrow[u, dashed, shift left, "S"] \end{tikzcd}$$

$$B\,[\vec{x}]_\beta = S^{-1}\big(A\big(S\,[\vec{x}]_\beta\big)\big)$$

so $\mathrm{col}_j\, B = \mathrm{col}_j\,(S^{-1}AS)$ (take $\vec{x} = \vec{v}_j$)

so $\boxed{B = S^{-1}AS}$ Method 1

OR, If $\vec{x} = \vec{v}_j$, $[\vec{v}_j]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow$ entry j

so $\boxed{\mathrm{col}_j(B) = B\,[\vec{v}_j]_{\mathcal{B}} = [f(\vec{v}_j)]_{\mathcal{B}}}$

Method 2

Example

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(same $\mathbb{R}^2$ basis as Monday notes)

$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Method 1

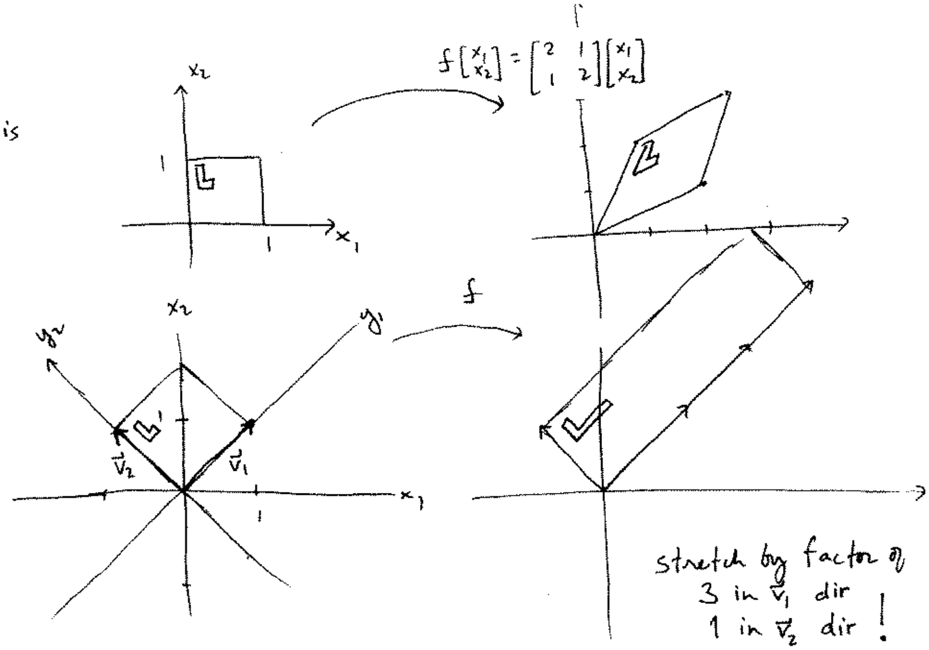
$B = S^{-1}AS$

Fill in both methods!

Method 2

$col_1(B) = [f(\vec{v}_1)]_{\mathcal{B}}$; $col_2(B) = [f(\vec{v}_2)]_{\mathcal{B}}$

This transformation $f(\vec{x})$ is better understood using $\mathcal{B}$!



<u>Definition</u> A and B are <u>similar matrices</u> if $\exists$ invertible S so that

$$B = S^{-1}AS$$

<u>example</u> $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

<u>notice</u>: If A and B are similar you can think of A as the matrix of $f(\vec{x}) = A\vec{x}$ with respect to the standard basis, and B as the matrix of the <u>same</u> transformation function, but with respect to basis given by the columns of S. Thus many <u>matrix</u> properties of A & B will turn out to be "similar"

<u>notice</u>: if $B = S^{-1}AS$ then $SBS^{-1} = A$ $[(S^{-1})^{-1}BS]$ so being similar is <u>symmetric</u> in A & B

also, if $B = S^{-1}AS$ and $C = T^{-1}BT$
then $C = T^{-1}S^{-1}AST = (ST)^{-1}A(ST)$ so C is similar to A, i.e.

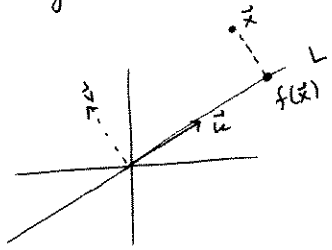
being similar is a <u>transitive</u> property

and $A = I^{-1}AI$ so A is similar to A;
being similar is <u>reflexive</u>

Any relationship between objects which is <u>reflexive</u>, <u>symmetric</u>, and <u>transitive</u> is called an <u>equivalence relation</u>, and allows you to partition your objects into subsets (equivalence classes) consisting of all mutually equivalent objects. [redacted]

Example: Every $\mathbb{R}^2$ projection matrix A is ~~equivalent~~ similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

proof.



Let $f(\vec{x})$ be projection onto L

Let $\vec{u}$ be a (unit) dir. vector for L

Let $\vec{v}$ be a (unit) $\perp$ vector to L

Let $\mathcal{B} = \{\vec{u}, \vec{v}\}$

Then $$[f]_{\mathcal{B}} = \left[[f(\vec{u})]_{\mathcal{B}} \;\middle|\; [f(\vec{v})]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example

What is a simple matrix that every <u>reflection matrix</u> must be similar to?

Example: Show $\underset{A}{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}$ is similar to $\underset{B}{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}$ and determine a and b.

$$S^{-1}AS = B$$

$$AS = SB$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x & s \\ y & t \end{bmatrix} = \begin{bmatrix} x & s \\ y & t \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{aligned} x + 2y &= ax \\ 4x + 3y &= ay \\ s + 2t &= sb \\ 4s + 3t &= tb \end{aligned}$$

$$\begin{aligned} (1-a)x + 2y &= 0 \\ 4x + (3-a)y &= 0 \\ s(1-b) + 2t &= 0 \\ 4s + t(3-b) &= 0 \end{aligned}$$

$$\left[\begin{array}{cc|cc|c} 1-a & 2 & 0 & 0 & 0 \\ 4 & 3-a & 0 & 0 & 0 \\ 0 & 0 & 1-b & 2 & 0 \\ 0 & 0 & 4 & 3-b & 0 \end{array}\right]$$

we want non-zero solutions $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} s \\ t \end{bmatrix}$, so 2×2 systems must <u>not</u> have inverse!

$$(1-a)(3-a) - 8 = 0 \qquad (1-b)(3-b) - 8 = 0$$

a, b are roots to

$$z^2 - 4z - 5 = 0$$
$$(z-5)(z+1) = 0$$

e.g. $a = 5$, $b = -1$

$$\underset{B}{\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}}$$

$$\begin{aligned} -4x + 2y &= 0 \\ 4x - 2y &= 0 \end{aligned}$$

e.g. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{aligned} 2s + 2t &= 0 \\ 4s + 4t &= 0 \end{aligned}$$

e.g. $\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

e.g. $S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$.