Math 2270-1
Monday 26 Sept.

Exam 1 tomorrow!

We will spend the 2^{nd} half of today's lecture going over the review sheet and selected questions on the practice exam.

But first introduce

§3.4 : Coordinates with respect to arbitrary subspace bases, ← today
and the matrix of a linear transformation with respect to domain & codomain bases
↑
Wed.

Recall, (page 6 Sept. 20), $\mathcal{B}$

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ be a basis for the subspace W. Then each $\vec{w} \in W$ is a unique linear combination of the basis vectors

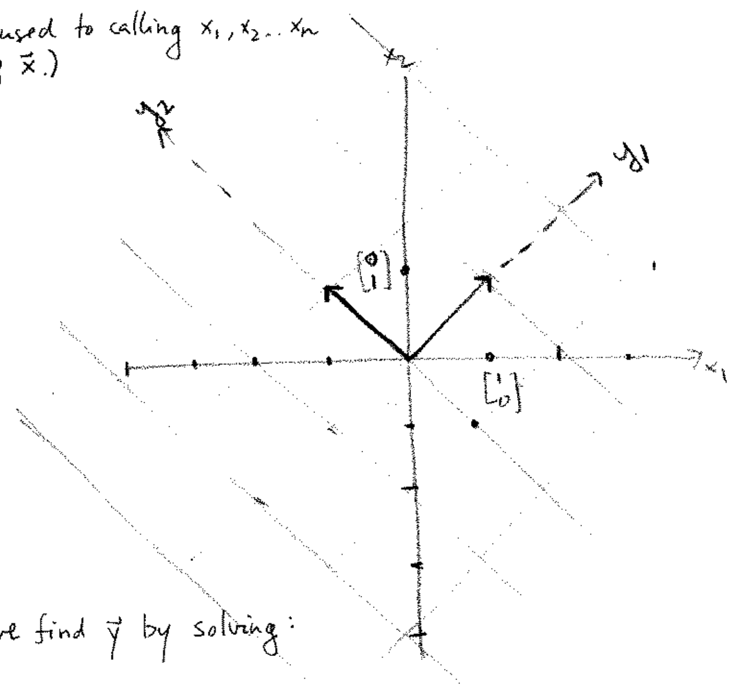
$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Do you remember why?

The vector $\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ ~~tect~~ is called the coordinate vector for $\vec{w}$ with respect to $\mathcal{B}$, $[\vec{w}]_{\mathcal{B}}$;
notice these linear combo coeffs tell you exactly how to get to $\vec{w}$ by traveling in the $\vec{v}_1, \ldots \vec{v}_k$ directions.

Example
Let $E = \{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$, the standard basis for $\mathbb{R}^n$
Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Then $[\vec{x}]_E = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$! (And you are used to calling $x_1, x_2, \ldots x_n$ the coords of $\vec{x}$.)



Example Let $W = \mathbb{R}^2$
$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(1) find $[\vec{x}]_{\mathcal{B}}$, and interpret geometrically
for $\vec{x} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Hint: If we write $[\vec{x}]_{\mathcal{B}} = \vec{y}$ then we find $\vec{y}$ by solving:

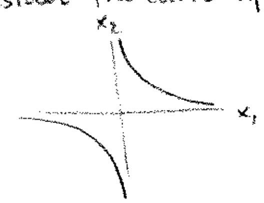
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[redacted]

3) Find $\vec{x} = [\vec{x}]_E$ if $[\vec{x}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $[\vec{x}]_B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

<u>Example cont'd</u>

Consider the curve $x_1 x_2 = 1$. What is its equation in the y_1-y_2 coord system?



Hint: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example Let W be the plane $x_1 + 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$
and $\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Find $[\vec{x}]_{\mathcal{B}}$ for $\vec{x} = \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} \in W$

(If W is a subspace and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is a basis for W
then for $\vec{w} \in W$ we find $[\vec{w}]_{\mathcal{B}}$ by solving the system.

$$\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_k \right] [\vec{w}]_{\mathcal{B}} = \vec{w}$$
)

Notes on the rank + nullity theorem
(these were discussed already in class).

<u>Theorem</u>: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear, $f(\vec{x}) = A_{m \times n}\vec{x}$
then $\dim(\ker(f)) + \dim(\operatorname{Im}(f)) = \dim(\text{domain}) = n$

<u>proof</u>.

$$A \longrightarrow \operatorname{rref}(A)$$

$$\begin{array}{cccccccc}
1 & 0 & x & 0 & x & 0 & 0 & x \\
0 & 1 & x & 0 & x & 0 & 0 & x \\
0 & 0 & 0 & 1 & x & 0 & 0 & x \\
\vdots & 0 & 0 & 0 & 0 & 1 & 0 & x \\
 & & & & \vdots & 0 & 1 & x \\
0 & 0 & 0 & & 0 & 0 & 0 & 0
\end{array}$$

$\dim(\operatorname{Im}(f)) = \#$ of leading 1's in $\operatorname{rref}(A)$
BECAUSE reducing A preserves column dependencies (& indepencies),
because each dependency
$$x_1 \operatorname{col}_1(A) + \dots + x_n \operatorname{col}_n(A) = \vec{0}$$
is exactly a soltn to $A\vec{x} = \vec{0}$ and these solutions don't change via elementary row ops.
THUS, the columns of A which correspond to cols of $\operatorname{rref}(A)$ without leading 1's are redundant and may be eliminated without reducing $\operatorname{span}\{\operatorname{col}_1(A), \dots \operatorname{col}_n(A)\}$
$= \operatorname{Image}(A)$.
The remaining cols are a basis of $\operatorname{Image}(A)$.

$\dim(\ker(f)) = \#$ of col's in $\operatorname{rref}(A)$ which do <u>not</u> have leading 1's (from rows).
BECAUSE this is the number of free parameters in the parametric expression of the solution set to $A\vec{x} = \vec{0}$ (i.e. $\vec{x} \in \ker A$),
AND when you write the general soltn to $A\vec{x} = \vec{0}$ as

$$* \qquad \vec{x} = t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k \qquad \text{by backsolving from } \operatorname{rref}(A)$$

and if t_j came from column n_j, then the n_j^{th} entry in $*$ is exactly t_j, so if you set $\vec{x} = \vec{0}$ you deduce $t_j = 0 \ \forall j$,
so $\vec{v}_1, \dots \vec{v}_k$ are independent!