Math 2270-1
9/20 Tuesday
Chapter 3 con't'd.

How do you represent a subspace explicitly?

rough answer : as the span of a good collection of vectors. : e.g. pages 2, 4. (Fri-Mon notes)

What makes a collection good?

precise notion is linear independence. See page 6

ans you don't want redundant vectors, that is ones which can be written as linear combinations of the others.

If you have dependent vectors you may discard them without reducing the span, as in page 2 example. This is a general fact:

If $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + d\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + d\left(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k\right)$$
$$\in \text{span}\{\vec{v}_1, \ldots \vec{v}_k\}$$

so $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k, \vec{w}\} = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$.
so you may delete $\vec{w}$ without reducing the span

Theorem If $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ span W, then a subset of $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is a basis for W.

proof: delete dependent vectors until your collection is independent. It will still span W. ∎

Definitions page - know these!

Def a <u>subspace</u> $W \subset \mathbb{R}^n$ is a subset which is closed under addition and scalar multiplication.

Def A <u>linear combination</u> of $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is any vector $\vec{v}$ which can be expressed as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$.

Def <u>span</u> $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$:= the set of linear combos, i.e.

$$\{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k : c_1, c_2, \dots c_k \in \mathbb{R}\}$$

Def <u>$\vec{v}$ is linearly dependent</u> on $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ means it is a linear combo, i.e. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$

Def <u>The set $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is linearly dependent</u> (if at least one $\vec{v}_j$ is dependent on the remaining $k-1$ vectors. The symmetric way of saying this is)

if some linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

where not all c_i's $= 0$.

Def $\{\vec{v}_1, \dots \vec{v}_k\}$ is <u>linearly independent</u> (if no $\vec{v}_j$ is a linear combo of the others) if the only way

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

is when $c_1 = c_2 = \dots = c_k = 0$.

Def A subset $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ of the subspace W is a <u>basis</u> for W if it <u>spans W</u> and is <u>linearly independent</u>.
[this is good]. In this case we say that <u>dimension $(W) = k$</u>

Theorem If $\{\vec{v}_1, \dots \vec{v}_k\}$ is a basis for W, then each $\vec{w} \in W$ has a <u>unique</u> representation $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$

proof: if also $\vec{w} = d_1\vec{v}_1 + \dots + d_k\vec{v}_k$ then $\vec{w} = \vec{w}$ so

$$(c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \dots + (c_k - d_k)\vec{v}_k = \vec{0}$$

so $c_1 - d = c_2 - d_2 = \dots = c_k - d_k = 0$!

Examples

1. The "standard basis" $\{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$ is a basis for $\mathbb{R}^n$

   Check!!!

How many other bases can you think of for $\mathbb{R}^n$?
Do they all have the same number of vectors?
(use rref!)

If $f(\vec{x}) = A\vec{x}$ we often say kernel of A and image of A
(rather than kernel of f or image of f)

Example (similar to #25 §3.3)

Find bases for the kernel of A, and then for image(A), where

```
> with(linalg):
> A:=matrix(4,5,
  [4,8,1,1,0,
  3,6,1,2,1,
  2,4,1,9,2,
  1,2,3,2,11]);
```

$$A := \begin{bmatrix} 4 & 8 & 1 & 1 & 0 \\ 3 & 6 & 1 & 2 & 1 \\ 2 & 4 & 1 & 9 & 2 \\ 1 & 2 & 3 & 2 & 11 \end{bmatrix}$$

```
> rref(A);
```

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A neat way to find a "better" basis of the image is to go to reduced column echelon form (which you can get from rref using transpose):

```
> transpose(rref(transpose(A)));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -10 & 15 & -2 & 0 & 0 \end{bmatrix}$$

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>
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