HW [illegible] Fri 9/23

3.1 3, 5, 6, 13, 17, [illegible], 23, [illegible]

3.2 1, 4, [illegible], 7, 8, [illegible], 15, 21, 24, 33, 45

3.3 6, 7, 18, 21, 25, 27, 32, 35, 37

technology!

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Math 2270-1
Friday Sept 16
§3.1

$f: X \to Y$

X is called the <u>domain</u>

Y is called the <u>co-domain</u> ← to avoid confusion about the meaning of "range"

$\{f(x) \in Y : x \in X\}$ is called <u>image(f)</u>

<u>example</u> $f: \mathbb{R} \to \mathbb{R}$

$f(x) := e^x$

domain $= \mathbb{R}$
codomain $= \mathbb{R}$
image$(f) = \{y \in \mathbb{R} \mid y > 0\}$.

We want to study <u>image(f)</u> for $f: \mathbb{R}^n \to \mathbb{R}^m$, <u>$f$ linear</u>

$f(\vec{x}) = A\vec{x}$

(1) $f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v})$
(2) $f(k\vec{u}) = kf(\vec{u})$
$\forall \vec{u}, \vec{v} \in \mathbb{R}^n$
$k \in \mathbb{R}$

We also want to study the <u>kernel</u> of f ($\ker(f)$)

$\ker(f) := \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = \vec{0}\}$

"vectors which get squashed"

<u>examples in $\mathbb{R}^2$</u>

	image	kernel
rotations		
reflections		
projections		

<u>example</u> $f: \mathbb{R}^n \to \mathbb{R}^n$, $f(\vec{x}) = A\vec{x}$ $A_{n \times n}$ A^{-1} exists <u>iff</u>

(1)
(2)
(3)
(4)
(5)

example $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -4 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$

ker(f) is easy

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 2 & -4 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$-R_1+R_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \quad \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2+R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = 2t$
$x_1 = -t$

$$\boxed{\ker f = \left\{ t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}}$$

line thru origin.

image(f) a little harder

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 2 & -4 & y_2 \\ 1 & 1 & -1 & y_3 \end{array}\right]$$

$$-R_1+R_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 2 & -4 & y_2 \\ 0 & 2 & -4 & y_3-y_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & y_1 \\ 0 & 1 & -2 & y_2/2 \\ 0 & 0 & 0 & y_3-y_1-y_2 \end{array}\right]$$

this will be solvable iff $-y_1-y_2+y_3=0$

i.e. iff $\vec{y}$ lies on the plane thru origin $\left\{\vec{y}\in\mathbb{R}^3 : y_1+y_2-y_3=0\right\}$

(or better!)

Image(f)

$$= \left\{ x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \text{all linear combos of } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \right\}.$$

$$:= \text{span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$$

What can I do to a collection of vectors to clean them up and figure out their span?

1. interchange 2 vectors
2. multiply a vector by non-zero const
3. replace a vector by its sum with a scalar multiple of another vector

expressing $f(\vec{x})$ in the linear combo of cols way, and looking at ker(f), we see

the ker(f) tells me about possible redundancy in my "span" expression for image(f)

$$-1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} = 0$$

so $\begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. so $\text{span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\} = \text{span}\left\{\vec{v}_1, \vec{v}_2\right\}$ is a plane!

$\vec{v}_3 = \vec{v}_2 - 2\vec{v}_1$

What kind of subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ can $\ker f$ and image(f) be when f is linear?

Theorem $\ker(f)$ is closed under addition and scalar multiplication

proof: We need to show that if $\vec{u}, \vec{v} \in \ker f$ and $k \in \mathbb{R}$, then

(1) $\vec{u} + \vec{v} \in \ker f$

(2) $k\vec{u} \in \ker f$

Theorem image(f) is closed under addition and scalar multiplication.

proof: We need to show that if $\vec{w}, \vec{z} \in$ image(f) & $k \in \mathbb{R}$ then

(1) $\vec{w} + \vec{z} \in$ image(f)

(2) $k\vec{w} \in$ image(f)

$\boxed{\text{So, what subsets of } \mathbb{R}^n \text{ are closed under addition and scalar multiplication? e.g. in } \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3\text{?}}$

possible subsets of $\mathbb{R}^3$ closed under addition and scalar multiplication
(let S be such a subset (called a <u>subspace</u>))

let $\vec{u} \in S$
then $0\vec{u} = \vec{0}$ is a scalar mult, so

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} \in S$$

one possibility [closed under + & s.m]

If this is not all of S, then let
$\vec{u} \neq \vec{0}$, $\vec{u} \in S$.

$$\Rightarrow \{t\vec{u}\} \subset S$$

another possibility: line thru origin

If $L = \{t\vec{u} : t \in \mathbb{R}\}$ is not all of S, let $\vec{v} \in S$, $\vec{v} \notin L$

$$\Rightarrow \{s\vec{u} + t\vec{v} : s, t \in \mathbb{R}\} \subset S$$

plane thru origin.
[could get normal vector by $\vec{u} \times \vec{v}$ or
rref $\left[\begin{array}{c|c}\vec{u} & 0\\ \vec{v} & 0\end{array}\right]$]

If the plane above is not all of S,
let $\vec{w} \in S$, $\vec{w} \notin$ plane.
then $\{s\vec{u} + t\vec{v} + r\vec{w} : s, t, r \in \mathbb{R}\} = \mathbb{R}^3$!

<u>proof</u>: this is "clear" geometrically but requires an algebraic proof.
The explanation is quite clever, with ideas <u>we shall use over & over</u>:

- Notice $s\vec{u} + t\vec{v} + r\vec{w} = \left[\vec{u} \,|\, \vec{v} \,|\, \vec{w}\right]\begin{bmatrix}r\\s\\t\end{bmatrix}$ can never equal to zero vector, unless $\begin{bmatrix}r\\s\\t\end{bmatrix} = \vec{0}$:

if $r \neq 0$, $s\vec{u} + t\vec{v} + r\vec{w} = \vec{0}$
$\Rightarrow \vec{w} = -\frac{s}{r}\vec{u} - \frac{t}{r}\vec{v}$,
which it isn't!
So $r = 0$. But then $s\vec{u} + t\vec{v} = 0$
and $t \neq 0 \Rightarrow \vec{v} = -\frac{s}{t}\vec{u}$
which it isn't!
So $t = 0$. Then $s\vec{u} = \vec{0}$
So $s = 0$ ∎

thus $\left[\begin{array}{c|c|c|c}\vec{u} & \vec{v} & \vec{w} & 0\\ & & & 0\\ & & & 0\end{array}\right]$ has only the zero sol

$\downarrow$ rref

<u>must be</u>
$$\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}$$
(else ∞'ly many sols!)

But this also means $\left[\begin{array}{c|c|c|c}\vec{u} & \vec{v} & \vec{w} & y_1\\ & & & y_2\\ & & & y_3\end{array}\right]$ always has a (unique) sol,
so our set IS $\mathbb{R}^3$!! ∎