Math 2270
Tuesday 13 Sept
§2.4 & fractal ideas

$\downarrow$

Matrix algebra:

<u>matrix addition</u> (for matrices of equal dimensions) $A=[a_{ij}]$, $B=[b_{ij}]$

$$\text{entry}_{ij}(A+B) := a_{ij}+b_{ij}$$ (this is just vector addition for strangely-shaped vectors)

<u>scalar multiplication</u>

$$\text{entry}_{ij}(kA) := k a_{ij}$$ (vector scalar mult.)

<u>matrix multiplication</u>

$B_{k\times m}$, $A_{m\times n}$ $\Rightarrow$ $BA_{k\times n}$

$$\text{entry}_{ij}(BA) := \text{row}_i(B)\cdot \text{col}_j(A).$$

<u>identity matrix</u> $I_{n\times n}$: $\text{col}_j(I)=\vec{e}_j$; i.e. $\text{entry}_{ij}(I)=\begin{cases}1 & i=j\\ 0 & i\neq j.\end{cases}$

Recall, we showed that if

$f(\vec{x})=A\vec{x}$
$g(\vec{y})=B\vec{y}$ then $g(f(\vec{x}))=B(A\vec{x})$ actually equals $(BA)\vec{x}$. $*$

<u>properties</u> (room on page 2 for work)

$AB\neq BA$ in general [don't expect $g(f(x))=f(g(x))$ for arbitrary fcns!]

$A(BC)=(AB)C$ this follows from our work on compositions of matrix transformations!

$$\begin{aligned}\text{col}_j[A(BC)] &= A\,\text{col}_j(BC)\\ &= A[B\,\text{col}_j C]\\ &= (AB)\,\text{col}_j C \quad \text{by } *\\ &= \text{col}_j[(AB)C]\end{aligned}$$

$A(B+C)=AB+AC$
$(A+B)C=AC+BC$

<u>check:</u>

If A, B have inverses (and have same dimension)
then $(AB)^{-1}=B^{-1}A^{-1}$

<u>check:</u>
(algebraic,
transformation)

if A^{-1} exists then the soltion to

$$A\vec{x} = \vec{b} \quad \text{is}$$

$$\vec{x} = A^{-1}\vec{b}$$

check:

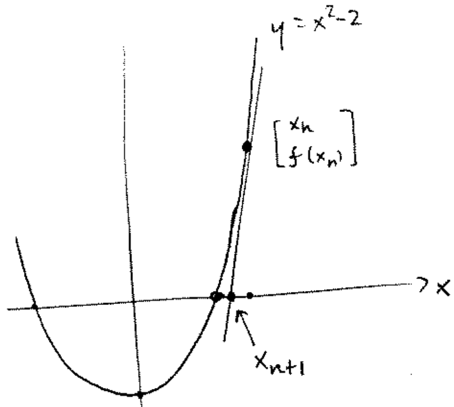
(algebraic,
transformation)

Ideas behind the fractal generation algorithm:

Analogy you might have seen in Calculus:

using iteration techniques and "contractions" to find $\sqrt{2}$ (Newton's method!)

- this is a root of $f(x) = x^2 - 2$



$$\frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n)$$

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ Newton iteration.

$$f(x) = x^2 - 2$$
$$f'(x) = 2x$$
$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$
$$\boxed{x_{n+1} = \tfrac{1}{2}\left(x_n + \tfrac{2}{x_n}\right)}$$

$$x_{n+1} - x_n = \frac{1}{2}\left[x_n + \frac{2}{x_n} - \left(x_{n-1} + \frac{2}{x_{n-1}}\right)\right]$$
$$= \frac{1}{2}\left[(x_n - x_{n-1}) + \frac{2(x_{n-1} - x_n)}{x_n x_{n-1}}\right]$$
$$= \frac{1}{2}\left(x_n - x_{n-1}\right)\left(1 - \frac{2}{x_n x_{n-1}}\right)$$

If $x_0 \geq 1$ then all $x_n \geq 1$, and

$$|x_{n+1} - x_n| \leq \tfrac{1}{2}|x_n - x_{n-1}|$$
$$\leq \tfrac{1}{4}|x_{n-1} - x_{n-2}|$$
$$\leq \tfrac{1}{2^n}|x_1 - x_0|$$

So the sequence $\{x_n\}$ converges! and the limit x satisfies

$$x = \tfrac{1}{2}\left(x + \tfrac{2}{x}\right)$$
$$\tfrac{1}{2}x = \tfrac{1}{x}$$
$$x^2 = 2 \ !$$

Abstract formulation:

Space: points in the interval $[1, 10]$

Contraction map:

$$T(x) := \tfrac{1}{2}\left(x + \tfrac{2}{x}\right)$$
$$T : [1,10] \longrightarrow [1,10]$$

$$T(x) - T(z) = \tfrac{1}{2}\left(x + \tfrac{2}{x}\right) - \tfrac{1}{2}\left(z + \tfrac{2}{z}\right)$$
$$= \tfrac{1}{2}\left[(x-z) + \tfrac{2}{x} - \tfrac{2}{z}\right]$$
$$= \tfrac{1}{2}\left[(x-z)\right]\left[1 - \tfrac{2}{xz}\right]$$

so

$$|T(x) - T(z)| \leq \tfrac{1}{2}|x - z|$$

T is a "contraction": image points are closer than domain points, by a strict factor $\mu < 1$.

($\mu = \frac{1}{2}$ in this example.)

For fractals:

Space: closed, bounded subsets of $\mathbb{R}^2$, using Hausdorff distance ("Bobs")

Contraction map:

$$\mathcal{F}(Bob) = f_1(Bob) \cup \cdots \cup f_k(Bob)$$

where each f_k contracts $\mathbb{R}^2$.

$\sqrt{2}$!

```
> Digits:=30;
  x:=10;
  for i from 1 to 10 do
  x:=.5*(x+2/x);
  od;
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$$Digits := 30$$

$$x := 10$$

$$x := 5.10000000000000000000000000000$$

$$x := 2.74607843137254901960784313726$$

$$x := 1.73719487437959832272787729873$$

$$x := 1.44423809486623193896379583829$$

$$x := 1.41452565514873774122499431255$$

$$x := 1.41421359680226932847147453155$$

$$x := 1.41421356237309546789256872287$$

$$x := 1.41421356237309504880168872421$$

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$$x := 1.41421356237309504880168872421$$

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> sqrt(2.0);
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$$1.41421356237309504880168872421$$

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>
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