Math 2270-1
Fall 2005
Practice Exam SOLUTIONS

1a) Show that the following matrix equation has no solution

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

(5 points)

I could write down the augmented matrix, row reduce, and show that I get an inconsistent equation 0=1. Or, for this simple system I can argue directly: he last equation says x_2=4, the second equation says x_1 =0, but then the first equation would say that 1(0)+2(4)=2, which is false.

1b) Find the least-squares solution to the problem in part (1a).

(10 points)

I solve the normal equation $A^T A\, x = A^T b$*:*

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2) Let V be the span of the vectors

$$\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

in R^3.

2a) Find an orthonormal basis for the plane V.

(10 points)

Use Gram-Schmidt. I'm in Maple, you would do this by hand.

```
> with(linalg):
> v1:=vector([1,-1,0]);
  v2:=vector([2,0,1]);
  w1:=1/sqrt(2)*v1;
     #first orthogonal element
  z2:=v2-(dotprod(v2,w1))*w1;
     #second orthogonal element;
  w2:=1/norm(z2,2)*z2;
     #second orthonormal element
  [evalm(w1),evalm(w2)];#orthonormal basis,
     #I used brackets because maple disorders
     #sets, i.e. {x,y}={y,x].
```

$$v1 := [1, -1, 0]$$

$$v2 := [2, 0, 1]$$

$$w1 := \frac{1}{2}\sqrt{2}\ v1$$

$$z2 := v2 - v1$$

$$w2 := \frac{1}{3}\sqrt{3}\ (v2 - v1)$$

$$\left[\left[\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}, 0\right], \left[\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right]\right]$$

So, writing things vertically as we are used to doing, this would read

$$\left\{\frac{1}{2}\sqrt{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \frac{1}{3}\sqrt{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

2b) Let $v = \begin{bmatrix}2\\0\\4\end{bmatrix}$. Find the projection of v onto the subspace V, using your answer from (2 a)

(10 points)

```
> v:=vector([2,0,4]);
  proj(v):=dotprod(v,w1)*w1+dotprod(v,w2)*w2;
  evalm(proj(v));
```

$$v := [2, 0, 4]$$

$$\text{proj}(v) := -v1 + 2\ v2$$

$$[3, 1, 2]$$

So the projection vector is $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

2c) If you did your computations correctly, your answer to part (2b) should equal the matrix from number (1) multiplied by your least squares solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to part (1b), i.e.

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix}.$$

Explain why.

(5 points)

Check:

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

The reason for this is that the least squares solution to an inconsistent matrix system Ax=b IS the solution to Ax=proj(b) onto the image of A. The image of A in #1 is the span of A's columns, which is the subspace W of #2. Thus Ax gives the projection vector worked out in #2b. (If you think about it, the method of least squares gives you a way of doing projection problems without first finding orthonormal bases.)

3a) Define a *linear transformation* L:V->W .

(5 points)

L is ***linear*** *means that for all u,v in V and all scalars c*
(A) L(u+v)=L(u)+L(v), and
(B) L(cu)=cL(u).

3b) Define the *kernel* and *image*, for a linear transformation L

(5 points)

kernel(L):={v in V such that L(v)=0}
image(L):={w in W such that w=L(v) for some v in V}

3c) Prove that the kernel of a linear transformation is a subspace.

(5 points)

We need to verify that kernel(L) is closed under addition and scalar multiplication:
(A) Closure under addition: Let u and v be in ker(L). This means L(u)=0, L(v)=0. Therefore,

L(u+v)=L(u)+L(v)=0, so u+v is also in the kernel of L.
(B) Closure under scalar multiplication: Let u be in ker(L). Then L(cu)=cL(u)=c0=0, so cu is in ker(L).

4) Let P2 be the space of polynomials in "t" of degree at most 2. Define T:P2->P2 by

$$T(f) = f'' + 4f.$$

4a) Find the matrix for T, with respect to the standard basis $\{1, t, t^2\}$.

(5 points)

Write B for the matrix of T with respect to our basis. The jth column of B is the coordinate vector for T of the jth basis element. So we compute:
T(1)=4 .... coord vector is [4,0,0]
T(t)=4t .....coord vector is [0,4,0]
T(t^2)=2+4t^2 ....coord vector is [2,0,4].
Thus

$$B := \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4b) Compute $T(c_0 + c_1 t + c_2 t^2)$ directly and then verify that the matrix you found in (4a) converts the coordinate vectors of inputs into coordinate vectors of outputs appropriately.

(5 points)

If we write $p(t) = c_0 + c_1 t + c_2 t^2$ *then* $p'(t) = c_1 + 2 c_2 t$ *and* $p''(t) = 2 c_2$, *so*

$$T(p(t)) = 2 c_2 + 4\, p(t)$$

$$T(p(t) = 2 c_2 + 4 c_0 + 4 c_1 t + 4 c_2 t^2$$

So the coordinates of T(p(t)) with respect to our basis are $\begin{bmatrix} 2 c_2 + 4 c_0 \\ 4 c_1 \\ 4 c_2 \end{bmatrix}$. *This should be the the matrix B times the coordinates of p(t), i.e.*

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

And it is!

4c) Find bases for image(T) and kernel(T).

(10 points)

p(t) is in the kernel of T if and only if the coordinates of T(p(t)) are zero, i.e. if and only if the coordinates are in the kernel of the matrix B above. Since rref(B) is clearly the identity, only zero is in the kernel, so only the zero polynomial is in the kernel. The basis of a zero dimensional subspace is *empty.*

p(t) is in the image of T if and only if its coordinates are in the image of the matrix B. Since the columns of B span all of R^3 it follows that the coordinate of the image polynomials are all possible coordinates, i.e. the image is all of P2 - so we may use our P2 basis $\{1, t, t^2\}$ a*s a basis for the image.*

4d) verify the rank+nullity theorem for T

(5 points)

rank + nullity = domain dimension.
3+0=3

5) Let

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

be a linear (matrix) map from R^2 to R^2. Let

$$B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$$

be a non-standard basis for R^2. Find the matrix for L with respect to the basis B.

(20 points)

Method 1:
Write v1=[1,1], v2=[-1,1]. The columns of the matrix are the B-coords of L(v1), and L(v2).

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Since the B-coords of [4,4] are [4,0], and the B-coords of [2,-2] are [0,-2], the matrix of L with respect to our non-standard basis is

$$\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

Method 2:
*Use transition matrices. If we call the matrix with respect to the nonstandard matrix B, then the transition matrix which converts B-coords to E-coords, S=**S**E<-B, just has the E-coords of the B basis in its columns, i.e.*

$$S := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The matrix with respect to the standard basis, A, was given in the problem:

$$A := \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

*The relation between A and B is (check!) B=inverse(S)*A*S:*

$$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

6) Find the least squares line fit for the following collection of 4 points: {[0,0],[1,2],[-1,-1],[-2,-3]}.
(10 points)

We seek the best approximate solution to

$$m \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}.$$

This is the matrix equation

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}.$$

The least squares solution solves

$$\begin{bmatrix} 0 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ \frac{3}{10} \end{bmatrix}$$

So the least squares line fit is y=1.6x + 0.3

7) Find the orthogonal complement in R^4 to the span of

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(10 points)

We want vectors perpendicular to the basis of V given above. This is exactly the kernel of

$$A := \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

$$RREF := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & \frac{-1}{2} & 0 \end{bmatrix}$$

Backsolving for the homogeneous equation we see x4=s, x3=t, x2=(1/2)t, x1=t-s, in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

So a basis for the orthogonal complement is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

8) True-False: 4 points for each problem; two points for the answer and two points for the reason.

(32 points)

8a) Any collection of 3 polynomials of degree at most 2 must be linearly dependent.
FALSE: P2 is 3 dimensional, so has lots of bases consisting of 3 vectors

8b) If the columns of a square matrix are orthonormal then so are the rows.
*TRUE: We discovered this magic in our discussion of orthogonal matrices ... if A has orthonormal columns, then transpose(A)*A=I, from which it follows that A*(transpose(A))=I, which says that the rows of A are orthonormal.*

8c) The formula

$$[AB]^T = A^T B^T$$

holds for all square matrices.
FALSE:

$$[AB]^T = B^T A^T$$

(and since A^T and B^T can be arbitrary, these two products will not be equal in general, because matrix multiplication is not commutative in general.

8d) If x and y are any two vectors in R^n, then $||x+y||^2 = ||x||^2 + ||y||^2$.
*FALSE: Using the dot product you see that ||x+y||^2 = ||x||^2 + ||y||^2 + 2*dotprod(x,y). You only get the Pythagorean Theorem when x and y are orthogonal.*

8e) If V is a 3-dimensional subspace of R^6, then the orthogonal complement to V is also 3-dimensional.
TRUE: In general the dimension of V plus the dimension of its orthogonal complement add up to the dimension of the Euclidean space. This is because if you take a basis for V and augment it with a basis for V perp, it is "easy" to show you have a basis for R^n.

8f) If A is symmetric and B is symmetric, then so is A+B
TRUE: if aij=aji and bij=bji then aij+bij=aji+bji

8g) The functions $f(t) = t^2$ and $g(t) = t^3$ are orthogonal, with respect to the inner product

$$< f, g > = \int_{-1}^{1} f(t)\, g(t)\, dt$$

TRUE. If you compute the inner product of f and g you get

$$\int_{-1}^{1} t^5\, dt = 0$$

8h) If a collection of vectors is dependent, then any one of the vectors in the collection may be deleted without shrinking the span of the collection.
FALSE: You may only delete a vector which is dependent on the remaining ones to ensure you don't shrink the span. For example, the following three vectors are dependent, but if you remove the third one you decrease the span:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$