Math 2270-1
Practice Exam SOLUTIONS

September 23, 2005

1) The following system of equations corresponds to the geometric configuration of three planes intersecting in a line:

$$2x + y + 3z = 2$$
$$x - y = 1$$
$$3x + 3y + 6z = 3$$

1a) Exhibit the augmented matrix corresponding to this system of three equations in three unknowns. (5 points)

$$A := \begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 6 & 3 \end{bmatrix}$$

1b) Find the reduced row echelon form of your matrix from part (1a), and use it to solve the system. (10 points)

$$RREFA := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Backsolving, we see that z=t, y=-t, x=1-t. In vector form this is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

1c) Verify that the direction vector for the line of intersection which you found in part 1b) is orthogonal (perpendicular) to each of the three plane normal vectors (5 points)

The direction vector is [-1,-1,1]. The normal to the first plane is [2,1,3]. The dot product of these two vectors is -2-1+3=0, so they are perpendicular. Continuing with the dot products between the line direction and the other two plane normals we get -1+1=0 and -3-3+6=0.

2) Consider the matrix

$$B := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

2a) Use elementary row operations to find the inverse matrix to B. Check your answer by verifying that B times its inverse yields the identity matrix.

(15 points)

We augment B with the identity matrix and then but this 3 by 6 matrix into rref - the last three columns will be the inverse of B, provided B reduced to the identity:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 1 & \frac{-1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{4} \end{bmatrix}$$

$$Binv := \begin{bmatrix} -2 & \frac{3}{4} & \frac{1}{4} \\ 1 & \frac{-1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{-1}{4} \end{bmatrix}$$

*then you should check that the matrix called Binv satisfies B*Binv=I.*

2b) Use your inverse matrix from 2b) or 2c) to solve the system:

(5 points)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$$

*The solution is x=Binv*b, i.e.*

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} -2 & \frac{3}{4} & \frac{1}{4} \\ 1 & \frac{-1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ -2 \end{bmatrix}$$

3) Give the two definitions (abstract and concrete) for a map (function) L from R^n to R^m to be linear. Explain why the definitions are equivalent.

(15 points)

The concrete definition is that L can be expressed as a matrix map, i.e. L(x)=Ax, where A is an m by n matrix. This is equivalent to L satisfying the two properties:

(i) L(u+v)=L(u)+L(v)

(ii) L(cu)=cL(u)

for all vectors u, v in R^n and all real scalars c.

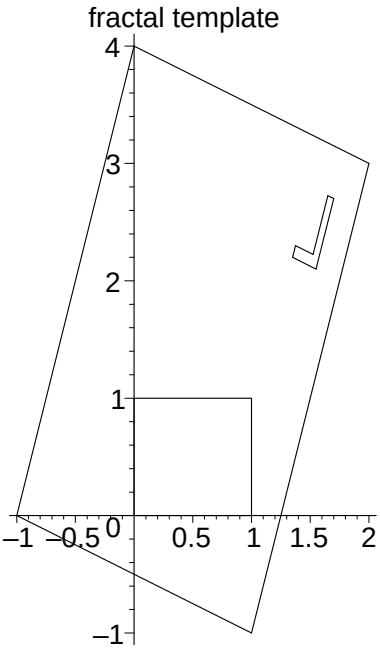
The proof that these definitions are equivalent is as follows:

(1) If L(x)=Ax, then since matrix multiplication satsifies A(x+y)=Ax+Ay and A(cx)=cA(x), we see that L satisfies (i) and (ii).

*(2) If L satisfies (i) and (ii) then it follows that L(c1*u1 + c2*u2 + ... + ck*uk) = c1*L(u1) + c2*L(u2) + ... + ck*L(uk). Hence L(x1*e1 + x2*e2 + ... + xn*en)= x1*L(e1) + x2*L(e2) + ... + xn*L(en). [Here we expand the vector x in terms of the standard basis vectors.] But, using the linear combination way of expressing a matrix times a vector, we see that L(x)=Ax where A is the matrix with columns L(e1), L(e2), ... L(en).*

) Write down the affine map which created the following "L-picture". (The parallelgram is the image of the unit square, by this affine map.)

(10 points)



f(x)= Ax+b. Hence f(0)=b. From the L-picture we see that f(0)=[1,-1], so this is the translation vector b. f(e1)=col1(A)+b. Since f(e1) is [-1,0] we see that col1(A) is f(e1)-b, i.e. the vector from [1,-1] to [-1,0], i.e. the vector [-2,1]. Similarly, the second column of A is the vector from [1,-1] to [2,3], i.e. [1,4]. Thus the affine map is given by

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5) Here is a matrix A:

$$A = \begin{bmatrix} 2 & -4 & -1 & 1 & -1 & 0 \\ 1 & -2 & -1 & 0 & -2 & -2 \\ 1 & -2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{bmatrix}$$

We consider the linear map, f(x)=Ax. Here is the reduced row echelon form of A:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5a) Find a basis for the image of f, which is a subset of the original six columns. Explain your reasoning.

(5 points)

*The image of f is spanned by the columns of A. We may read their linear dependencies off from the dependencies of the columns of rref(A). (Since dependencies correspond to solutions of the homogeneous equation, and so remain the same as we do row operations to A (hence to a augmented with the zero vector.) Thus column 2 of A is -2*column 1 of A, and columns 4,5,6 are dependent on columns 1 and 3. Columns 1 and 3 are independent. Since removing dependent vectors from a set does not decrease the span of the set we deduce that columns 1 and 3 of A are a basis for the image of f:*

$$\left\{\begin{bmatrix}2\\1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}\right\}$$

5b) Express the sixth column of A as a linear combination of the basis vectors you found in part 5a.

(5 points)

We figure out the dependency from rref(A), as above. Thus

$$\begin{bmatrix}0\\-2\\2\\4\end{bmatrix}=2\begin{bmatrix}2\\1\\1\\0\end{bmatrix}+4\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}$$

5c) Find a basis for the kernel of f.

(10 points)

We backsolve from rref(A): x6=t, x5=s, x4=r, x3=-r-3s-4t, x2=u, x1=2u-r-s-2t. In vector form this is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=t\begin{bmatrix}-2\\0\\-4\\0\\0\\1\end{bmatrix}+s\begin{bmatrix}-1\\0\\-3\\0\\1\\0\end{bmatrix}+r\begin{bmatrix}-1\\0\\-1\\1\\0\\0\end{bmatrix}+u\begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix}$$

The four exhibited vectors are linearly independent (If you look at entries 2,4,5,6 you will see immediately that the only linear combo of them which gives the zero vector is when all the coefficients are zero), so they are a basis for the nullspace:

$$\left\{\begin{bmatrix}2\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\-3\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\-4\\0\\0\\1\end{bmatrix}\right\}$$

5e) State the theorem which relates the the dimensions of image(f), kernel(f) to the matrix dimensions. Verify that this theorem holds for the matrix A above. Explain why this theorem holds for every matrix. (10 points)

dim(image(f)) + dim(ker(f)) = dim(domain(f)) = n.
In our case this equality is 2+4=6.

6) True-False: 2 points for the right answer and 2 points for the justification, on each part: (40 points total)

6a) If A and B are square matrices, then

$$(A-B)(A+B)=A^2-B^2$$

False*: when you expand (A-B)(B-A), using the fact that multiplication distributes over addition, you get*

$$[A-B][A+B]=A^2-BA+AB-B^2$$

This doesn't equal A^2-B^2, unless AB=BA, which is not true in general.

6b) The following identity is true, for invertible matrices A and B:
inverse(AB) = inverse(B)*inverse(A).

True:

$$A\,B\,B^{[-1]}\,A^{[-1]}=A\,A^{[-1]}$$

and

$$A\,A^{[-1]}=I$$

6c) If the matrix product AB = 0 (where 0 is the zero matrix), and if B is non-singular, then A must be the zero matrix.

True*: nonsingular means the inverse matrix exists: Thuse we may multiply the equation AB=0 on both sides by the inverse of B (on the right side), i.e. AB(inverse(B))=0(inverse(B)=0, which simplifies to A=0.*

6d) If A is a square matrix and A*A=A, then A=the identity matrix.

False*. If A was invertible this was true, but projection matrices also satisfy A^2=A. For example, projection onto the x-axis is given by the matrix*

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

6e) If Ax=0 is a homogeneous system with an m by n matrix, and if the number of rows ''m'' is less than the number of columns ''n'', then there are always infinitely many solution vectors x.

True: *homogeneous equations are always consistent, so solutions will exist. There will be infini9tely*

many because there will be columns without leading 1's in rref(A), so there will be free parameters in the general solution .

6f) If 2u+3v+4w=5u+6v+7w then the subspace spanned by {u,v,w} is at most 2-dimensional.

True: *From that equation we can express w as a linear combination of u and v, for example. Thus the subspace is spanned by two vectors (and possibly one or zero!) So it is at most 2-dimensional.*

6g) If AB=0 then BA=0 as well. (A and B are square matrices)

False: For example you could take

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

6h) Any three linearly independent vectors in R^3 are actually a basis for R^3.

True: *They span a 3-dimensional space, and 3-dimensional subspaces of R^3 msut be all of R^3 (otherwise we could find 4 linearly independent vectors in R^3.)*

6i) If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

True: *linear dependencies on the columns of A correspond to solutions of the homgeneous equation Ax=0 (when we write the matrix product Ac in linear combo form). So saying that the only solution to Ac=0 is the zero vector is exactly saying that the only linear comb of the columns which adds up to zero is the one in which all the coefficients are zero.*

6j) If the vectors u,v,w are linearly dependent then w is a linear combination of v and w:

False: *for example the set of vectors*

$$\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$