Name SOLUTIONS
I.D. number

Math 2270-2
Final Exam
December 13, 2001

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 200 points possible, and the point values for each problem are indicated in the right-hand margin. Of course, this exam counts for 30% of your final grade even though it is scaled to 200 points. Good Luck!

1) Let

$$A := \begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -4 & 1 & 3 \\ -1 & 2 & 1 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Then T(**x**):=A**x** is a linear a map from **R**^4 to **R**^4. Here are the reduced row echelon forms of A and of the transpose of A:

$$\mathrm{RREF}(A) := \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{RREF}(A^T) := \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1a) Find a basis for the kernel of T.

(10 points)

backsolve $\mathrm{rref}(A)|0$:

$x_4 = t$
$x_3 = t$
$x_2 = s$
$x_1 = 2s - 2t$

$$\vec{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

1b) Find a basis for the image of T.

(5 points)

$A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3 \middle| \vec{v}_4 \right]$ We see from rref(A) that

$\vec{v}_2 = -2\vec{v}_1$

$\vec{v}_4 = 2\vec{v}_1 - \vec{v}_3$

$\vec{v}_1, \vec{v}_3$ independent

$\Rightarrow \{\vec{v}_1, \vec{v}_3\}$ a basis, $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(could also use the 1st two rows of rref(A^T))

1c) Express the 4th column of A as a linear combination of the two non-zero rows in the reduced row echelon form of A transpose.

(5 points)

$$\begin{bmatrix} 2 \\ 3 \\ -3 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -3 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\leftarrow$ matched 1st 2 entries by inspection

Note: row ops on A^T correspond to col ops on A & so preserve the column space. That's why we know we can express v_4 in terms of the non-zero rows of rref(A^T)

1d) For a subspace V of R^n, define what is meant by the orthogonal complement to V.

(5 points)

$$V^{\perp} := \{ x \in \mathbb{R}^n \text{ s.t. } \vec{x} \cdot \vec{v} = 0 \ \forall \vec{v} \in V \}$$

1e) For the matrix A on page 1, find a basis for the orthgonal complement to the kernel of A. What do we usually call this subspace, which is one of the four fundamental subspaces associated to the matrix?

(5 points)

$\ker(A) \perp$ rowspace(A) (they are orthog complements)

(by defintion of $A\vec{x} = \vec{0}$!)

rowspace basis $= \{[1, -2, 0, 2], [0, 0, 1, -1]\}$

(non-zero rows of rref(A))

Since row ops preserve the rowspace

1f) For our matrix A on page 1, find a basis for the orthogonal complement to the image of A. What is another name for this subspace?

(10 points)

$= (\text{colspace}(A))^\perp$

$= (\text{rowspace}(A^T))^\perp$

$= \ker(A^T)$

from rref(A^T):

$y_4 = t$

$y_3 = s$

$y_2 = -s - t$

$y_1 = 3s + 2t$

$$\vec{y} = s\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

basis $\left\{\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

2a) Exhibit the rotation matrix which rotates vectors in R^2 by an angle of α radians in the counter-clockwise direction.

(5 points)

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

2b) Verify that the product of an α-rotation matrix with a β-rotation matrix is an $(\alpha+\beta)$-rotation matrix.

(7 points)

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

2c) Use Euler's formula to expand both sides of the identity

$$\mathbf{e}^{(i\alpha)}\,\mathbf{e}^{(i\beta)} = \mathbf{e}^{(i(\alpha+\beta))}$$

and verify that (as in 2b) this identity is equivalent to the addition angle formulas for cos and sin.

(8 points)

$$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$$= \cos(\alpha+\beta) + i(\sin(\alpha+\beta))$$ iff addition angle formulas hold.

$$= e^{i(\alpha+\beta)}$$

3) Let V be the vector space of polynomials in t of degree at most 1. Let T:V->V be defined by

$$T(f) := 3\,D(f) - 2f$$

where D stands for t-derivative.

3a) Show that T is linear.

(5 points)

$$\text{(a)}\ T(f+g) = 3(f+g)' - 2(f+g)$$
$$= 3f'-2f + 3g'-2g$$
$$= T(f)+T(g)$$

$$\text{(b)}\ T(cf) = 3(cf)' - 2(cf)$$
$$= c(3f'-2f)$$
$$= cT(f)$$

3b) Let

$$\beta = \{1, t\}$$

be a basis for V. Find the matrix B for T with respect to this basis.

(10 points)

$$T(1) = -2 \qquad [T(1)]_\beta = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$T(t) = 3 - 2t \qquad [T(t)]_\beta = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\text{so}\quad B = \begin{bmatrix} -2 & 3 \\ 0 & -2 \end{bmatrix}$$

3c) Let

$$\Gamma = \{1+t, 1-t\}$$

be a different basis for V. Find the matrix S which converts Gamma-coordinates to beta coordinates. In other words, S times the Gamma-coordinates of a vector yields the beta-coordinates.

(5 points)

$$S_{\beta \leftarrow \Gamma} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ [1+t]_\beta & [1-t]_\beta \end{array}$$

3d) Use the matrix S from part 3c and its inverse, together with the matrix B from part 3b, in order to find the matrix G for T with respect to the basis Gamma. (Or, for only 5 points, find this matrix another way.)

(10 points)

$$G = S_{\Gamma\leftarrow\beta} \, B \, S_{\beta\leftarrow\Gamma} = S^{-1}BS$$

$$= -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -1/2 & -3/2 \\ 3/2 & -7/2 \end{bmatrix}$$

check: $T(1+t) = 3-2(1+t)$
$= 1-2t$
$= -\frac{1}{2}(1+t)+\frac{3}{2}(1-t)$ ✓

$T(1-t) = -3-2(1-t)$
$= -5+2t$
$= -\frac{3}{2}(1+t)$
$-\frac{7}{2}(1-t)$ ✓

4) Let

$$A := \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

4a) If you have studied carefully you know how to find the least squares solution to Ax=b, by solving the system

$$A^T A\, x = A^T b$$

Carry out his procedure to find the least squares problem for A and b as above.

(5 points)

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 1 & 1 \\ \hline 1 & 1 & 2 \\ 0 & -1 & -3 \end{array}\right] \quad -2R_1+R_2$$

$x_2 = 3$
$x_1 = -1$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}}$$

4b) The least squares solution is the vector x so that Ax equals the projection of b onto the image of A. Use this characterization to explain the derivation of the displayed formula in part 4a.

(5 points)

let $\vec{x}^*$ be l.s. soltn.

then $A\vec{x}^* = \text{proj}_V b \qquad V = \text{colspace}(A)$

$\Leftrightarrow \quad A\vec{x}^* - \vec{b} \in V^\perp$

$\Leftrightarrow \quad A\vec{x}^* - \vec{b} \perp \text{colspace}(A)$

$\Leftrightarrow \quad A\vec{x}^* - \vec{b} \perp \text{rowspace}(A^T)$

$\Leftrightarrow \quad A^T(A\vec{x}^* - \vec{b}) = 0$

$\Leftrightarrow \quad A^TA\vec{x}^* = A^T\vec{b}$

4c) Find an orthonormal basis for the column space (=image) of A.

(10 points)

$$\boxed{\left\{ \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}}$$

$\vec{v}_1 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$

$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$

$\vec{z}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1$

$= \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} // \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad \vec{w}_2 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

4d) Find the projection of b onto the column space of A, using your basis from 4b.

(5 points)

$\text{proj}_V b = (\vec{b}\cdot\vec{w}_1)\vec{w}_1 + (\vec{b}\cdot\vec{w}_2)\vec{w}_2 \quad ; \quad \vec{b} = \begin{bmatrix}1\\2\\3\\0\end{bmatrix}$

$= \underbrace{\frac{2}{2}\frac{1}{2}}_{1/2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \underbrace{\frac{6}{2}\frac{1}{2}}_{3/2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$

4e) Verify that the projection of b which you found in 4c does indeed equal Ax, where x is the least squares solution you found in 4a.

(5 points)

$\begin{bmatrix}1 & 1\\-1 & 0\\1 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$ ✓

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5) This is a discrete dynamical system story problem: Company "B" invents a new, neat internet application, which can be downloaded from their company page for a cost of \$10.00. Company "M", reverse engineers the product, and begins including the application as a free part of their dominant browser package. Let x(t) represent the fraction of computers using B's application or no application at all, and let y(t) represent the fraction of computers using M's version. Suppose that initially, x(0)=1, y(0)=0.

Suppose that the following transition equation describes how these fractions change month by month:

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

5a) Explain the meaning of the transition equations above: how do customers change applications from month to month? A diagram may help.

(5 points)

$$\begin{tikzcd} B \arrow[r, bend left, ".2"] & M \arrow[l, bend left, ".05"] \end{tikzcd}$$

$x(t+1) = .8\,x(t) + .05\,y(t)$

frac .2 leave category B each mo., for M

.05 frac leave M for B

$y(t+1) = .2\,x(t) + .95\,y(t)$

get 20% of B's

95% retained, 5% lost

5b) Find the eigenvalues and eigenvectors for the transition matrix above. Hint: this is a regular transition matrix so lambda=1 will be an eigenvalue. This should help you factor the characteristic polynomial.

(20 points)

$$\begin{vmatrix} \lambda-.8 & -.05 \\ -.2 & \lambda-.95 \end{vmatrix} = \lambda^2 - 1.75\lambda + .75$$
$$= (\lambda-1)(\lambda-.75)$$

$$\begin{array}{r} \overset{4}{.95} \\ \times .8 \\ \hline .760 \\ -.01 \\ \hline .75 \end{array}$$

$\lambda = 1$:

$$\left[\begin{array}{cc|c} .2 & -.05 & 0 \\ -.2 & .05 & 0 \\ \hline 20 & -5 & 0 \\ 0 & 0 & 0 \\ \hline 4 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\lambda = .75$

$$\left[\begin{array}{cc|c} -.05 & -.05 & 0 \\ -.2 & -.2 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5c) Using the eigenvalues and eigenvectors from part 5b find a closed form expression for

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(5 points)

if $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ $\quad v_1, v_2$ evects

then $A^t\vec{x}_0 = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1/5 \\ -4/5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so $\vec{x}_0 = \frac{1}{5}\begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A^t\vec{x}_0 = \frac{1}{5}\begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{4}{5}(.75)^t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5d) What fractions will x(t) and y(t) be close to 12 months later?

(5 points)

$(.75)^{12}$ is nearly 0, so

$$A^{12}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} \begin{matrix} \leftarrow x(12) \\ \leftarrow y(12) \end{matrix}$$

6a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

$$* \qquad x^T A\, x + B\, x + c = 0$$

into one without any ''cross terms''. Be precise in explaining the change of variables, and the justification for why such a change of variables exists.

(5 points)

A is symmetric so there is an orthog mat S st
$S^T A S = D$ (diagonal $\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

(by spectral thm)

so let $\vec{x} = S\vec{u}$ in * and get

$$u^T \underbrace{S^T A S}_{D} \vec{u} + B S \vec{u} + c = 0$$

$$\sum \lambda_i u_i^2 + B S \vec{u} + c = 0$$

(complete square to finish the problem)

6b) Apply the procedure from part (4a) to put the conic section (this is a curve in R^2)

$$5\,x^2 + 6\,x\,y - 3\,y^2 = 24$$

into standard form. Identify the conic section and sketch it in the x-y plane (use the next page), showing the rotated axes.

(20 points)

$$[x, y] \begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 24$$

$$|\lambda I - A| = \lambda^2 - 2\lambda - 24$$
$$= (\lambda - 6)(\lambda + 4)$$

$\lambda = 6$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ -3 & 9 & 0 \\ \hline 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = -4$

$$\left[\begin{array}{cc|c} -9 & -3 & 0 \\ -3 & -1 & 0 \\ \hline 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

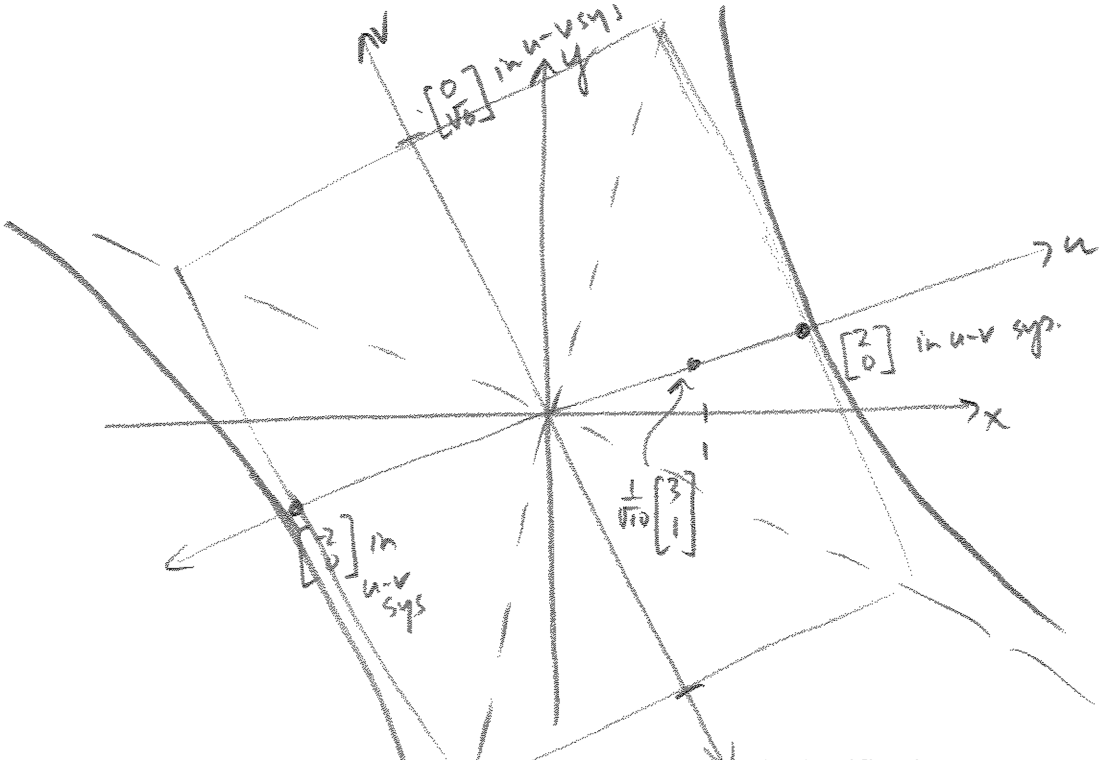
$\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

$$S = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$6u^2 - 4v^2 = 24$$

$$\frac{u^2}{4} - \frac{v^2}{6} = 1 \quad \text{hyperbola!}$$



7) True-False. Four points each (2 points for answer, two points for justification.)

(20 points)

7i) Let A be a nonsingular n by n matrix. Then the equation Ax=Ay implies that x=y.

T $\quad Ax = Ay$

$\Rightarrow A^{-1}Ax = A^{-1}Ay \Rightarrow x = y$

7ii) In the discrete dynamical system problem #5 above, if the initial customer distribution had included a positive fraction who already used the company M software (so that y(0)>0), then the limiting market share for company M would have been even greater.

F. $\quad$ the $.75^t$ term always $\to 0$ as $t \to \infty$ leaving 1:4 ratio of B:M customers

7iii) If A and B are symmetric (square) matrices (of the same size) then so is their product AB.

F $\quad \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}3 & 3\\5 & 4\end{bmatrix}$

7iv) Let A be a rectangular matrix whose columns are orthonormal. Then

$$A^T A = I$$

where I is the identity matrix.

T $\quad row_i(A^T) \cdot col_j(A) = col_i(A) \cdot col_j(A) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

7v) There exists a matrix with three rows and four columns which has the property that the columns are orthonormal.

F $\quad$ cannot have 4 o.n. vectors in $\mathbb{R}^3$ since they'd be lin. ind.