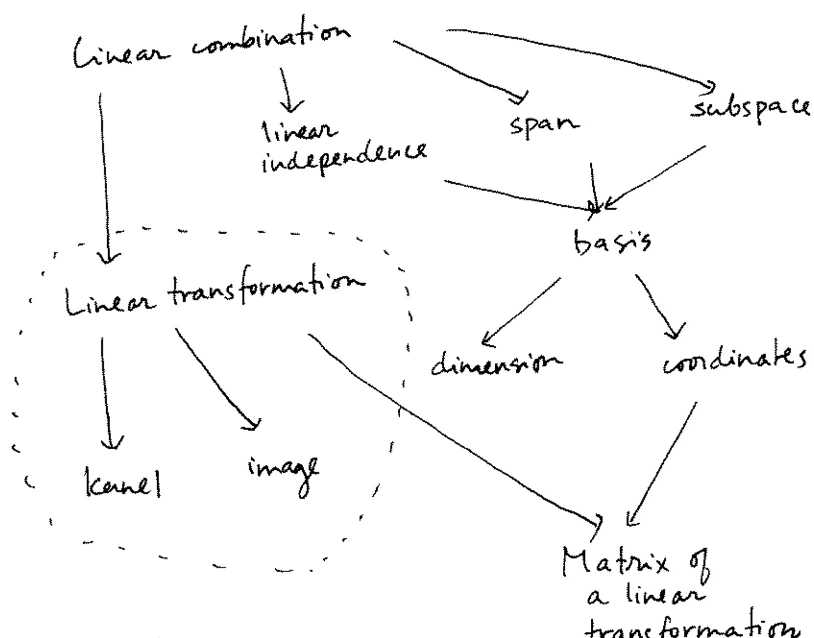
Math 2270
Tuesday 10/4

- Finish pages 3-4 Monday notes

Other theorems whose proofs carry over to general linear (combination) spaces:

Theorem: Let $\dim(V) = n < \infty$, V a vector space.
Any collection $\{f_1, \ldots f_k\}$ of independent vectors, $k < n$, can be successively augmented with vectors (not in the current span) to create a basis
$$\{f_1, \ldots f_k, g_{k+1}, \ldots g_n\}$$

Theorem: Any spanning set $\{f_1, \ldots f_N\}$ for the linear space V can be successively culled of redundant vectors to create a basis.



Def: If V, W are linear spaces, and $L: V \to W$ is a function with domain V, codomain W. Then L is linear provided

(a) $L(f+g) = L(f) + L(g)$ $\forall f, g \in V$
(b) $L(kf) = k\,L(f)$ $\forall f \in V, k \in \mathbb{R}$.

equivalent def

(c) $L(k_1 f + k_2 g) = k_1 L(f) + k_2 L(g)$ $\forall f, g \in V, k \in \mathbb{R}$.

<u>Def</u>: If $L: V \to W$ is a <u>linear</u> transformation

ker(L) (kernel of L) $:= \{ f \in V \text{ s.t. } L(f) = 0 \}$
im(L) (image of L) $:= \{ g \in W \text{ s.t. } g = L(f) \text{ some } f \in V \}$

<u>Example</u>: If $\{f_1, .. f_n\} = \mathcal{B}$ is a basis for V,
then we checked yesterday that the coordinate tranformation

$$L(f) := [f]_{\mathcal{B}}$$

$$L: V \longrightarrow \mathbb{R}^n$$

is linear.

What is ker(L)?

What is image(L)?

<u>Def</u>: If V, W are linear spaces, $L: V \to W$ a linear transformation,
then L is an <u>iso</u> <u>morphism</u>
($\downarrow$ equal, $\downarrow$ structure)

provided

$$\text{kernel}(L) = \{0\} < V$$
$$\text{image}(L) = W$$

notice this means
L is <u>1-1</u>, i.e. $L(f_1) = L(f_2)$ iff $f_1 = f_2$
$\Leftrightarrow L(f_1 - f_2) = 0 \Leftrightarrow f_1 - f_2 = 0$
($\ker L = \{0\}$)

and L is <u>onto</u>, i.e. image(L) = W.

Thus the inverse function
$L^{-1}: W \to W$ exists.
L^{-1} itself is linear, <u>and</u> an isomorphism! check!

Isomorphisms let you study problems in V by studying the "equivalent" problem in W, like we did Monday, using the coordinate transformation

<u>Example</u> : $L : \mathbb{P}_2 \to \mathbb{P}_2$

$L(f) = f'$

so, $L(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$

- L is linear (you know this from Calc!, $(f+g)' = f' + g'$
  $(cf)' = cf'$

ker(L) =

image(L) =

<u>Theorem</u> If $L : V \to W$ is linear then

1) ker(L) is a subspace of V
2) image(L) is a subspace of W

<u>proof:</u> (do you remember??)

Theorem (rank + nullity!) Let $L : V \to W$ linear, $\dim(V) = n < \infty$.

Def $\text{nullity}(L) := \dim(\ker L)$
$\text{rank}(L) := \dim(\text{image}(L))$

Then $\text{rank}(L) + \text{nullity}(L) = \dim(V)$.

Examples : $L(f) = f'$ page 3.
$L(f) = [f]_{\mathcal{B}}$, coordinate transformation.

proof: (this one looks a little new).

Let $\dim(V) = n$.
Let $\{f_1, \dots f_k\} \subset V$ a basis for $\ker(L)$
augment it to a basis
$\{f_1, \dots f_k, f_{k+1}, \dots f_n\}$ for V.

Then $L(c_1 f_1 + \dots + c_k f_k + c_{k+1} f_{k+1} + \dots + c_n f_n)$
$= 0 + c_{k+1} L(f_{k+1}) + \dots + c_n L(f_n)$.

So $\{L(f_{k+1}), \dots L(f_n)\}$ span $\text{image}(L)$.

They are also independent!

since $c_{k+1} L(f_{k+1}) + \dots + c_n L(f_n) = 0$

$\Rightarrow L(c_{k+1} f_{k+1} + \dots + c_n f_n) = 0$

$\Rightarrow c_{k+1} f_{k+1} + \dots + c_n f_n \in \ker(L)$

$\Rightarrow c_{k+1} f_{k+1} + \dots + c_n f_n = d_1 f_1 + \dots + d_k f_k$

$\Rightarrow$ all c_j's, d_i's $= 0$ since $\{f_1 \dots f_n\}$ lin ind. ∎

So $\dim(\ker(L)) = k$
$\dim(\text{image}(L)) = n - k$

∎