

Math 2270-1

Monday 10/31 Boo!

(1)

• 3x3 det's; Friday notes!

$n \times n$ determinants: We define $\det(A_{n \times n})$ recursively, in terms of $(n-1) \times (n-1)$ det's:

$$\det A := \sum_{i=1}^n (-1)^{i+1} \det(A_{i,1})$$

where $A_{i,1}$ is obtained from A by deleting row i & column 1.

We will prove many determinant properties using the principle of

Mathematical Induction:

Let $\{\text{statement}_n\}$ be a sequence of statements, indexed by natural numbers $n=1, 2, \dots$

If

(1) statement_1 is true

and if

(2) $\text{statement}_{(n+1)}$ can be proven assuming statement_n ,
(and if necessary, all statement_k , $k \leq n$)

then every statement in the sequence is true!

principle makes sense!

(1) \Rightarrow statement_1 true

(2) \Rightarrow $\text{statement}_2 \Rightarrow \text{statement}_3 \Rightarrow \dots$ all statements!

(You can "prove" the principle of mathematical induction using the axiom that any non-empty set of natural numbers has a least element.)

Example: prove $\sum_{i=1}^n i := (1+2+\dots+n) = \frac{1}{2}n(n+1)$

(Of course, b.y.o. Gauss saw the bigger picture of how to prove this without induction!)

(1)

(2)

The key fact, which we will prove by induction on Tuesday, is that you can expand a determinant down any column or across any row:

Theorem:
$$\det(A) = \sum_{i=1}^n (-1)^{i+j} \det(A_{i,j}) = \sum_{j=1}^n (-1)^{i+j} \det(A_{i,j})$$

 any $1 \leq j \leq n$ any $1 \leq i \leq n$.

Using this theorem, lots of other interesting results follow!

e.g.

Theorem: Let A be upper or lower triangular.

Then $\det(A) = a_{11} a_{22} \dots a_{nn}$
 is the product of the diagonal elts.

$$A = \begin{bmatrix} a_{11} & & & \\ x & a_{22} & & \\ x & x & \ddots & \\ x & x & x & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & x & x & x \\ & a_{22} & x & x \\ & & \ddots & x \\ 0 & & & a_{nn} \end{bmatrix}$$

- proof: ① True when $n=1$!
 ② Assume true for all $n \times n$ matrices.

Let $A_{(n+1) \times (n+1)}$ be upper triangular.

Expand down 1st column!

$$\det(A) = a_{11} \det(A_{1,1}) + \underbrace{\sum_{i=2}^n (-1)^{i+1} 0 \cdot \det(A_{i,1})}_{=0} = a_{11} \det(A_{1,1})$$

↓
 $a_{22} \dots a_{n(n+1)}$
 since $A_{1,1}$ is $n \times n$! □

Let $A_{(n+1) \times (n+1)}$ be lower triangular.

Expand across 1st row!

$$\det(A) = a_{11} \det(A_{1,1}) + \underbrace{\sum_{j=2}^n (-1)^{1+j} 0 \det(A_{1,j})}_{=0} = a_{11} \det(A_{1,1})$$

↓
 $a_{22} \dots a_{n(n+1)}$
 since $A_{1,1}$ is $n \times n$! □