Math 2270-1
Monday 10/31 Boo!

• 3×3 dets; Friday notes!

$n\times n$ determinants: We define $\det(A_{n\times n})$ recursively, in terms of $(n-1)\times(n-1)$ dets:

$$\det A := \sum_{i=1}^{n} (-1)^{i+1} \det(A_{i,1})$$

where $A_{i,1}$ is obtained from A by deleting row i & column 1.

We will prove many determinant properties using the principle of
<u>Mathematical Induction</u>:
Let $\{\text{statement}_n\}$ be a sequence of statements, indexed by natural numbers $n=1,2,\ldots$

If
(1) statement_1 is true
and if
(2) $\text{statement}_{(n+1)}$ can be proven <u>assuming</u> statement_n,
(and if necessary, all statement_k, $k\le n$)

then every statement in the sequence is true!

principle makes sense!
(1) ⇒ statement_1 true
(2) ⇒ statement_2 ⇒ statement_3 ⇒ ... all statements!
(You can "prove" the principle of mathematical induction using the axiom that any non-empty set of natural numbers has a least element.)

<u>Example</u>: prove $\sum_{i=1}^{n} i := (1+2+\cdots+n) = \frac{1}{2}n(n+1)$

(Of course, 6 y.o. Gauss saw the bigger picture of how to prove this without induction!)

(1)

(2)

The key fact, which we will prove by induction on Tuesday, is that you can expand a determinant down any column or across any row:

Theorem: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij})$

any $1 \le j \le n$ any $1 \le i \le n$.

Using this theorem, lots of other interesting results follow!

e.g.

Theorem: Let A be upper or lower triangular.

Then $\det(A) = a_{11} a_{22} \cdots a_{nn}$ is the product of the diagonal elts.

$$A = \begin{bmatrix} a_{11} & & & 0 \\ \times & a_{22} & & \\ \times & \times & \ddots & \\ \times & \times & \cdots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \times & \times & \times \\ & a_{22} & \times & \times \\ & & \ddots & \times \\ 0 & & & a_{nn} \end{bmatrix}$$

proof: (1) True when $n=1$!

(2) Assume true for all $n \times n$ matrices.

Let $A_{(n+1)\times(n+1)}$ be upper triangular.

Expand down 1st column!

$$\det(A) = a_{11} \underbrace{\det(A_{1,1})}_{\downarrow} + \underbrace{\sum_{i=2}^{n} (-1)^{i+1} \overset{= a_{i1}}{0} \cdot \det(A_{i,1})}_{0}$$

$a_{22} \cdots a_{(n+1)(n+1)}$

since $A_{1,1}$ is $n \times n$! $\blacksquare$

Let $A_{(n+1)\times(n+1)}$ be lower triangular.

Expand across 1st row!

$$\det(A) = a_{11} \underbrace{\det(A_{1,1})}_{\downarrow} + \underbrace{\sum_{j=2}^{n} (-1)^{1+j} \overset{a_{1j}}{0} \det(A_{1,j})}_{0}$$

$a_{22} \cdots a_{n+1,n+1}$

since $A_{1,1}$ is $n \times n$! $\blacksquare$