Math 2270-1
Monday 10/3
§ 4.1-4.3

Recall the abstract idea of a <u>linear space</u> (vectorspace).
And the tree we will follow in developing these ideas
- Friday notes page 1.

We had an example to do from the last page of Friday's notes, here it is repeated and expanded:

<u>Example</u> Let $\mathbb{P}_2 := \{ p(x) = a_0 + a_1 x + a_2 x^2 \text{ s.t. } a_i \in \mathbb{R} \} \subset \mathbb{F}(\mathbb{R}, \mathbb{R})$
(you can define $\mathbb{P}_n$, the subspace of polynomials of degree $\leq n$ similarly)

- Show $\mathbb{P}_2$ is a <u>subspace</u> of $\mathbb{F}(\mathbb{R}, \mathbb{R})$

- Is the collection of polynomials of degree <u>exactly</u> two a subspace of $\mathbb{F}(\mathbb{R}, \mathbb{R})$?

- Show $\underbrace{\{1, x, x^2\}}_{= \mathcal{B}}$ is a <u>basis</u> of $\mathbb{P}_2$

- Is $\{1+x,\ 1+x^2,\ x+x^2\} = \mathcal{C}$ another basis of $\mathbb{P}_2$?

- Let $p(x) = x^2$.
  Find the coords $[p(x)]_{\mathcal{B}}$, also $[p(x)]_{\mathcal{C}}$

- Is $\{1+x,\ 1+x^2,\ x+x^2,\ 1+x+x^2\}$ a basis for $\mathbb{P}_2$?

Hint: Use the basis $\mathcal{B} = \{1, x, x^2\}$ and take coordinates of the linear dependence/independence equation

$$c_1(1+x) + c_2(1+x^2) + c_3(x+x^2) + c_4(1+x+x^2) = 0$$

[redacted]

On page 2 we used the important fact that taking coordinates is a "linear" operation:

<u>Theorem</u>: Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ be a basis for $V$. Then $\forall \vec{u}, \vec{v} \in V$, $k \in \mathbb{R}$

$$[\vec{u} + \vec{v}]_{\mathcal{B}} = [\vec{u}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}}$$

$$[k\vec{u}]_{\mathcal{B}} = k[\vec{u}]_{\mathcal{B}}$$

<u>proof</u>:

<u>Example</u> : Find a basis for $\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ s.t. } a_{ij} \in \mathbb{R},\ \begin{matrix} 1 \le i \le 2 \\ 1 \le j \le 2 \end{matrix} \right\}$.

- Let $W \subset \mathbb{R}^{2\times 2}$, $W := \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ s.t. } a_{12} + a_{21} = 0 \right\}$

  - Show $W$ is a subspace of $\mathbb{R}^{2\times 2}$

  - Find a basis for $W$.

[redacted]

Def: the linear space $V$ has (finite) dimension $n$ iff $V$ has a basis $\mathcal{B} = \{f_1, f_2, .. f_n\}$ with $n$ elements

It is a general principle that if $V$ is $n$-dim'l then when you translate questions about vectors into statements about their $\mathcal{B}$-coordinates, then it's just like working in $\mathbb{R}^n$:

for example

Thm If $\dim(V) = n < \infty$, then any collection of $N > n$ vectors is dependent

proof: Let $S = \{g_1, g_2, .. g_N\}$ $N > n$

A linear combo

$$c_1 g_1 + c_2 g_2 + \dots + c_N g_N = 0$$

iff the analogous coordinate statement is true ( $\mathcal{B} = \{f_1, f_2 \dots f_n\}$ )

$$[c_1 g_1 + c_2 g_2 + \dots + c_N g_N]_{\mathcal{B}} = [0]_{\mathcal{B}}$$

iff $$\left.c_1 [g_1]_{\mathcal{B}} + c_2 [g_2]_{\mathcal{B}} + \dots + c_N [g_N]_{\mathcal{B}} = \vec{0} \quad \right\}$$ this is a homogeneous system of $n$ eq'ns with $N$ unknowns, $N > n$, so has non-trivial sol's.

Thm If $\dim(V) = n < \infty$ then no collection of $N < n$ vectors can span $V$

proof: Let $S = \{g_1, g_2, .. g_N\}$ $N < n$, $\mathcal{B} = \{f_1, f_2, .. f_n\}$.

Can we always find $x_1, x_2 .. x_N$ s.t.

$$x_1 g_1 + x_2 g_2 + \dots + x_N g_N = f = b_1 f_1 + b_2 f_2 + \dots + b_n f_n \quad ?$$

iff $[\quad]_{\mathcal{B}} = [\quad]_{\mathcal{B}}$

iff $$x_1 [g_1]_{\mathcal{B}} + x_2 [g_2]_{\mathcal{B}} + \dots + x_N [g_N]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A_{n \times N} \vec{x} = \vec{b}$$ can't be solved $\forall \vec{b}$ because $n > N$

( the span of the $[g]$-coords in $\mathbb{R}^n$ is at most $N$ dim'l!)

Exam 1 Score Distribution
n=29