Math 2270 ~ Friday Oct 28

- §5.4 HW due Tuesday (before class)
- MAPLE due Tuesday @ 5:00 p.m.
- Exam next Friday to cover thru chapter 5

- Discuss Wednesday notes on the 4 fundamental subspaces of linear maps $L: \mathbb{R}^n \to \mathbb{R}^m$.

- Begin determinants. (an important number associated to each square matrix.

$\det([a_{11}]) := a_{11}$ ← determines if the number has a multiplicative inverse!

$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} := a_{11}a_{22} - a_{21}a_{12}$ ← determines if matrix is invertible (iff $\det A \neq 0$).

then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

3×3 Case : $\det \left[\vec{u} \middle| \vec{v} \middle| \vec{w} \right] := \vec{u} \cdot (\vec{v} \times \vec{w})$ ← $\perp \vec{v}$ & $\vec{w}$, with magnitude $\|\vec{v}\|\|\vec{w}\|\sin\theta$ (2210)

$= 0$ iff $\vec{u} \perp (\vec{v} \times \vec{w})$

iff $\vec{u}, \vec{v}, \vec{w}$ lie in a common plane! (or line)

iff $\text{span}(\vec{u}, \vec{v}, \vec{w}) \neq \mathbb{R}^3$

iff $\text{Image}(A) \neq \mathbb{R}^3$

iff A^{-1} doesn't exist

$\vec{v} \times \vec{w} = $ "$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$" $= \begin{bmatrix} v_2w_3 - w_2v_3 \\ -v_1w_3 + w_1v_3 \\ v_1w_2 - w_1v_2 \end{bmatrix}$

$$\begin{aligned}
&\vec{u} \cdot (\vec{v} \times \vec{w}) \\
&= u_1 (v_2w_3 - w_2v_3) \\
&\quad + u_2 (-v_1w_3 + w_1v_3) \\
&\quad + u_3 (v_1w_2 - w_1v_2) \\
&= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}
\end{aligned}$$

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

← straight lines means take det (use brackets for matrix)

$$= \underbrace{u_1}_{a_{11}} \underbrace{\det(A_{11})}_{\text{cross out row 1 \& col 1}} - \overbrace{u_2}^{a_{21}} \underbrace{\det(A_{21})}_{\text{cross out row 2, col 1}} + \underbrace{u_3}_{a_{31}} \underbrace{\det(A_{31})}_{\text{cross out row 3 col 1.}}$$

Theorem If A is 3×3, $A = [a_{ij}]$. Let A_{ij} be the 2×2 matrix obtained by deleting row i & col j

then $\det A = \sum_{i=1}^{3} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} \det(A_{ij})$

expansion down col j, any $1 \le j \le 3$

expansion across row i, any $1 \le i \le 3$

– all 6 expansions give the same value (and $\det A \ne 0$ iff A^{-1} exists).

check 2 expansions (too tedious to check all 6!)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\left[(-1)^{i+j}\right] = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix} =$$

Magic:

Write $C_{ij} = (-1)^{i+j}\det(A_{ij})$ "cofactor"

$$\text{since } \det A = \sum_{i=1}^{3} \overset{\text{factor}}{a_{ij}}\,\overset{\text{cofactor}}{c_{ij}} = \sum_{j=1}^{3} a_{ij} c_{ij}$$

any j any i

Encode the determinant expansions
use the transpose of the cofactor matrix.

↓ fill in.

$$\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} |\vec{u}|\vec{v}|\vec{w}| & |\vec{v}|\vec{v}|\vec{w}| & \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} \det A & & \\ & & \\ & & \end{bmatrix}$$

try it first with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

$$[c_{ij}] = [(-1)^{i+j}\det[A_{ij}]] = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} -3 & 4 & 2 \\ & & \\ & & \end{bmatrix}$$

$$[c_{ij}]^T A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

So what is A^{-1}?