Math 2270-1
Wed 26 Oct

- Finish Fourier series, Tuesday notes.
- leftover topic (extra!):

$L: \mathbb{R}^n \to \mathbb{R}^m$

$L(\vec{x}) = A\vec{x}$

FOUR Fundamental subspaces

kernel(A)	image(A)
rowspace(A)	kernel(A^T)
orthogonal complements in domain	orthogonal complements in codomain

$V \subset \mathbb{R}^n$ subspace

$V^\perp = \{\vec{z} \in \mathbb{R}^n \text{ s.t. } \vec{z}\cdot\vec{v} = 0 \ \forall \vec{v} \in V\}$

each $\vec{x} \in \mathbb{R}^n$ can be expressed uniquely as

$\vec{x} = \vec{v} + \vec{z}$

where $\vec{v} \in V$, $\vec{z} \in V^\perp$.

In fact, $\vec{v} = \text{proj}_V \vec{x}$

$(V^\perp)^\perp = V$:

Let $\vec{x} \in (V^\perp)^\perp$.

Write $\vec{x} = \vec{v} + \vec{z}$ as above.

Since $\vec{x}$ is orthog to $V^\perp$,

$\vec{x}\cdot\vec{z} = 0$
$\quad \| $
$\vec{z}\cdot\vec{z}$

So $\vec{x} = \vec{v}$, $v \in V$,

and such an x is in $(V^\perp)^\perp$ ∎

<u>example</u> (from Fri Oct 21 notes)

$$L(\vec{x}) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ker A$: $A\vec{x} = \vec{0}$

$x_3 = t$
$x_2 = -t$
$x_1 = 3t$

$\ker(A) = \text{span}\left\{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\right\}$ orthog complements.

$\text{rowspace}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ why?

$= \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

$\text{image}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$ why

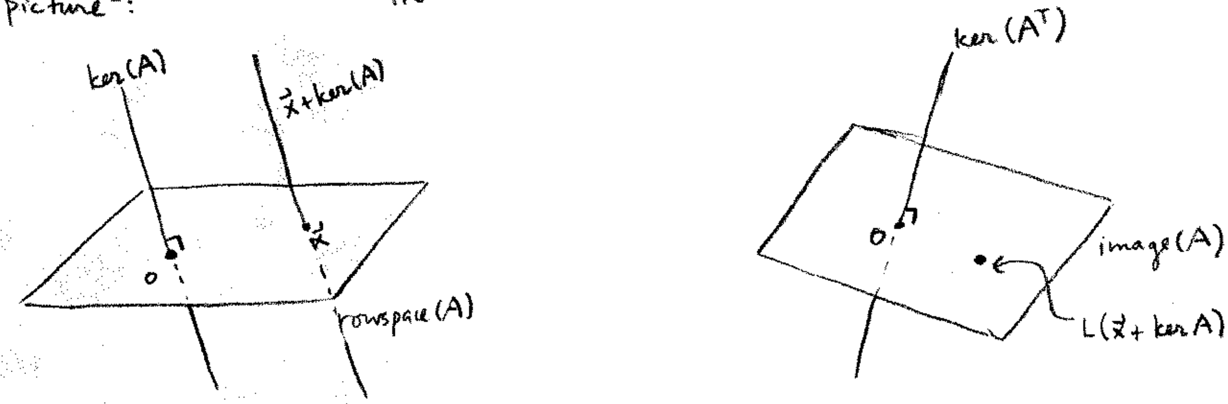
$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$

$\ker(A^T)$: $y_3 = s$, $y_2 = 2s$, $y_1 = -s$ $\quad \vec{y} = s\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\ker(A^T) = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right\}$

mapping picture:

$$\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$$



bigger example:

$L: \mathbb{R}^5 \to \mathbb{R}^4$, $L(\vec{x}) = A\vec{x}$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 3 & 2 & -2 & 1 & -1 \\ 1 & 2 & 0 & -3 & -7 \\ 0 & -2 & -1 & 5 & 10 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -\frac{5}{2} & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Find bases for the four fdl subspaces

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \text{rcef}(A) = 〃^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \end{bmatrix}$$

b) Verify that the two domain spaces are orthogonal complements
〃 〃 〃 codomain 〃 〃 〃 〃

Theorem Let $V \subset \mathbb{R}^n$. Then $\dim(V) + \dim(V^\perp) = n$

proof: Let $\{\vec{v}_1, \ldots \vec{v}_k\}$ a basis for V

Let $\{\vec{z}_1, \ldots \vec{z}_\ell\}$ a basis for $V^\perp$

since each $\vec{x} \in \mathbb{R}^n$ can be written as $\vec{x} = \vec{v} + \vec{z}$ $\quad \vec{v} \in V$, $\vec{z} \in V^\perp$

the set $\{\vec{v}_1, \ldots \vec{v}_k, \vec{z}_1, \ldots \vec{z}_\ell\}$ spans $\mathbb{R}^n$

now check independence:

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k + d_1\vec{z}_1 + \ldots + d_\ell\vec{z}_\ell = \vec{0}$

$\Rightarrow \text{proj}_V(\ldots) = \text{proj}_V \vec{0} = \vec{0}$

$\Rightarrow c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$

$\Rightarrow c_1 = c_2 = \ldots = c_k = 0 \longrightarrow d_1\vec{z}_1 + \ldots + d_\ell\vec{z}_\ell = \vec{0}$

$\Rightarrow d_1 = d_2 = \ldots = d_\ell = 0$ ✓

So $\{\vec{v}_1, \ldots \vec{v}_k, \vec{z}_1, \ldots \vec{z}_\ell\}$ is a basis for $\mathbb{R}^n$, so $k + \ell = n$

Corollary $L: \mathbb{R}^n \to \mathbb{R}^m$

$\dim(\text{rowspace}(A)) = \dim(\text{colspace}(A))$ [since if nullity$(L) = k$ then both of these other dims are $n-k$]

(In fact, L restricted to the rowspace(A) is an isomorphism to image(A) = colspace(A))

Could you prove this, for fun?