Math 2270-1
Tuesday 10/25 : Inner product spaces.

Inner product spaces...

Here's the flow chart of how we developed concepts from the dot product in $\mathbb{R}^n$:

$\mathbb{R}^n$ dot prod $\vec{x}\cdot\vec{y} := \sum_{i=1}^{n} x_i y_i$

algebra

$\vec{x}\cdot\vec{x} \geq 0. \quad \vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$

$\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$

$\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$

$(s\vec{x})\cdot\vec{y} = s(\vec{x}\cdot\vec{y}) = \vec{x}\cdot(s\vec{y})$

From algebra...

magnitude

orthogonal

Pythagorean Theorem

orthonormal basis (def) for subspace V

Gram-Schmidt construction

projection; $\mathbb{R}^n = V \oplus V^\perp$
i.e. $\vec{x} = \underbrace{\text{proj}_V \vec{x}}_{\in V} + \underbrace{\vec{z}}_{\in V^\perp}$, uniquely.

$\longrightarrow \text{proj}_{\vec{u}} \vec{v} \longrightarrow$ Cauchy Schwarz $\longrightarrow$ angle $\measuredangle \vec{u}, \vec{v}$

nearest point to $\vec{x}$ in V is $\text{proj}_V \vec{x}$

least squares sol'n to $A\vec{x} = \vec{b}$

data fitting.

An <u>inner product space</u> is a vector space V together with a inner product $\langle \cdot,\cdot \rangle$ which gives a real number for each pair of vectors. The inner product should satisfy

a. $\langle f,f\rangle \geq 0. \quad \langle f,f\rangle = 0$ iff $f = 0$

b. $\langle f,g\rangle = \langle g,f\rangle$

c. $\langle f, g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$

d. $\langle sf, g\rangle = s\langle f,g\rangle = \langle f, sg\rangle$

$\forall f,g \in V, \ s \in \mathbb{R}$

Since inner product spaces satisfy the same 4 algebra axioms, the entire concept tree at lower left also holds!!
(except maybe "least squares sol'ns" in case $\dim V = \infty$).

example let $V = \{ f : [a,b] \to \mathbb{R}$ s.t. f is piecewise continuous and integrable $\}$

$$\langle f, g \rangle := \int_a^b f(t) g(t)\, dt$$

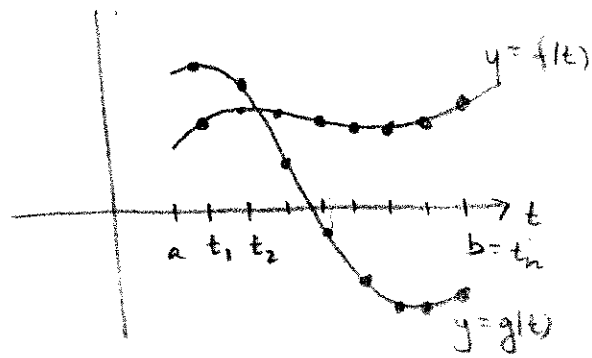
a.b.c.d. follow from integral properties

check:

This is not so different from the $\mathbb{R}^n$ dot prod, if you think of Riemann Sums

Let $\Delta t = \frac{b-a}{n}$

$t_i = a + i \Delta t$



$$\langle f, g \rangle = \int_a^b f(t) g(t)\, dt \approx \sum_{i=1}^{n} f(t_i) g(t_i) \Delta t$$

$$= (\Delta t) \begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_n) \end{bmatrix}$$

function inner product

$\mathbb{R}^n$ dot prod

take $\lim\limits_{n \to \infty}$ to get function dot prod!

example 1 : see part b of this week's Maple project ...

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$$

$P_0 \ P_1 \ P_2$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

Let's find an orthonormal basis for span$\{1, t, t^2\}$ by hand [in the handout you let MAPLE do the work] ...

example 2

$$\langle f,g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt \qquad \text{on } V = \{f:[-\pi,\pi]\to\mathbb{R},\ f \text{ piecewise continuous}\}$$

this example is also part of your Maple project

Theorem Let $V_n = \text{span}\{\frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots \cos nt, \sin t, \sin 2t, \ldots \sin nt\}$

Then the exhibited collection of $2n+1$ fcns is in fact an orthonormal basis for V_n

check it!!

You may want to use trig identities

$$\cos(m+k)t = \cos mt\cos kt - \sin mt\sin kt$$
$$\sin(m+k)t = \cos mt\sin kt + \sin mt\cos kt$$

whence!

$$\cos mt\cos kt = \tfrac{1}{2}[\cos(m+k)t + \cos(m-k)t]$$
$$\sin mt\sin kt = \tfrac{1}{2}[\cos(m-k)t - \cos(m+k)t]$$
$$\cos mt\sin kt = \tfrac{1}{2}[\sin(m+k)t + \sin(-m+k)t]$$

for $f \in V$

$\text{proj}_{V_n} f$ is called the n^{th}-order Fourier approximation to f.

$$\| \quad \frac{a_0}{2} + b_1 \sin t + b_2 \sin 2t + \dots + b_n \sin nt + c_1 \cos t + c_2 \cos 2t + \dots + c_n \cos nt$$

by formulas we already know, (since we have an orthonormal basis)!

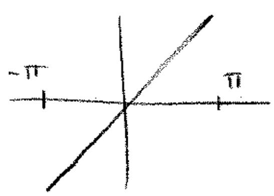
$$b_j = \langle \sin jt, f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin jt \, dt$$
$$c_j = \langle \cos jt, f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos jt \, dt$$

and the const term is

$$\langle \tfrac{1}{\sqrt{2}}, f \rangle \frac{1}{\sqrt{2}} = \frac{1}{2}\int_{-\pi}^{\pi} f(t)\,dt \qquad \left(\text{so } a_0 = \int_{-\pi}^{\pi} f(t)\,dt\right)$$

the a_0, $\{b_j\}$, $\{c_k\}$ are called Fourier coefficients

<u>example</u> $f(t) = t$



$a_0 = 0!$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t \overset{dv}{\overbrace{\sin kt}}}_{u} \, dt$$

$$= \frac{2}{\pi}\int_0^{\pi} t \sin kt \, dt \qquad \text{[integrand is an even function]}$$

$$= \frac{2}{\pi}\left[(t)\left(-\frac{\cos kt}{k}\right)\Big]_0^{\pi} - \underbrace{\int_0^{\pi} -\frac{\cos kt}{k}\,dt}_{\underset{0}{\|}} \right]$$

integrates to sine's, goes away

$$= \frac{2}{\pi}\frac{\pi}{k}(-1)^{k+1}$$

$$= \frac{2}{k}(-1)^{k+1}$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{t \cos kt}_{\text{odd fcn}} \, dt = 0!$$

$$\text{proj}_{V_n} \text{“}t\text{”} = 2\left[\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \dots + \frac{(-1)^{n+1}\sin nt}{n}\right]$$

> Use even-odd ideas in Maple, to do Fourier coeffs for $f(t) = |t|$

See Maple picture when $n = 10$, in your project notes.

Amazing fact is that

$$\| f - \mathrm{proj}_{V_n} f \| \to 0 \qquad \text{for } \underline{\text{any}}\ f \in V.$$

[too hard to show here].

but $f = \mathrm{proj}_{V_n} f + \underbrace{(f - \mathrm{proj}_{V_n} f)}_{\in V_n^\perp}$

$$\Rightarrow \|f\|^2 = \|\mathrm{proj}_{V_n} f\|^2 + \|f - \mathrm{proj}_{V_n} f\|^2$$

$$\|\mathrm{proj}_{V_n} f\|^2 = \left(\frac{a_0}{\sqrt{2}}\right)^2 + \sum_{i=1}^{n} b_i^2 + \sum_{j=1}^{n} c_j^2$$

as $n \to \infty$, deduce from "amazing fact" that

$$\|f\|^2 = \frac{a_0^2}{2} + \sum_{i=1}^{\infty} b_i^2 + \sum_{j=1}^{\infty} c_j^2$$

for $f(t) = t$

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{2}{3}\pi^2$$

$$\text{RHS} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

So $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$ "magic"

(ask Maple

sum(1/n^2, n=1..infinity);

see what happens.