Math 2270-1
Wed Oct 19

Maple Lab Friday - LCB 115.
Get me your height-weight data, if you haven't already!

Recall

Definition of orthogonal transformation $T(\vec{x}) = A\vec{x}$
(isometry).

Is the composition of two orthogonal transf's orthogonal?
What about the inverse?

What is the condition on the columns of A which is equivalent to $T(\vec{x}) = A\vec{x}$ being an orthogonal transformation?

Do page 4 of Tuesday notes, about transpose of a matrix, and the fact that A is orthogonal iff $A^{-1} = A^T$.

Some further algebra about transpose

$$(A^T)^T = A$$

$$(AB)^T = B^T A^T \quad \text{(notice order!)}$$

check this:

$$(A^{-1})^T = (A^T)^{-1}$$

check this:

<u>Def</u> $A = [a_{ij}]$ is <u>symmetric</u> iff $A^T = A$ (iff $a_{ij} = a_{ji}\ \forall i,j$)

$A = [a_{ij}]$ is <u>antisymmetric</u> iff $A^T = -A$ $(a_{ij} = -a_{ji})$

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix}$$

for any $A_{m\times n}$, $A^T A = A^T_{n\times m} A_{m\times n}$ is $n\times n$ symmetric

$A A^T = A_{m\times n} A^T_{n\times m}$ is $m\times m$ symmetric.

check:

The matrix for projection onto subspace $V \subset \mathbb{R}^n$

Let $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k\}$ be an orthonormal basis for V

Let $A = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \\ | & | & & | \end{bmatrix}$ $\qquad A_{n \times k}$

$$\text{proj}_V \vec{x} = (\vec{x} \cdot \vec{w}_1)\vec{w}_1 + (\vec{x} \cdot \vec{w}_2)\vec{w}_2 + \cdots + (\vec{x} \cdot \vec{w}_k)\vec{w}_k$$

We can write this in matrix form:

$$\text{proj}_V \vec{x} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \begin{bmatrix} \vec{x} \cdot \vec{w}_1 \\ \vec{x} \cdot \vec{w}_2 \\ \vdots \\ \vec{x} \cdot \vec{w}_k \end{bmatrix}$$

$$= \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix} \begin{bmatrix} \text{---} \; \vec{w}_1 \; \text{---} \\ \text{---} \; \vec{w}_2 \; \text{---} \\ \vdots \\ \text{---} \; \vec{w}_k \; \text{---} \end{bmatrix} \begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix} \qquad = (AA^T)\vec{x}$$

__example__ Let V be the plane $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : -x_1 + 2x_2 + x_3 = 0 \right\}$. Find the matrix for projection (wrt the standard basis) this new way, (AA^T) and the old way (change of basis)

initial basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

G.S. $\vec{w}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{z}_2 = \vec{v}_2 - \text{proj}_{V_1} \vec{v}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1$$

$$\vec{z}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\sqrt{2}}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

B = matrix wrt $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

the matrix we want is the composition

$$S_{E \leftarrow B} \, B \, S_{B \leftarrow E}$$

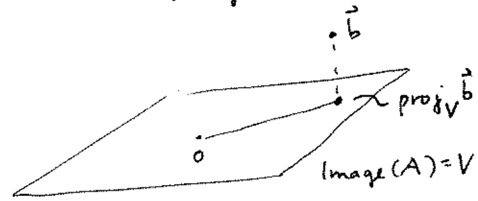
$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & -1/3 & 5/6 \\ 1/3 & 1/3 & -1/3 \\ -1/6 & 1/3 & 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \checkmark$$

(honest!)

§5.4 The least-squares approximate sol'n to $A\vec{x} = \vec{b}$

If $A\vec{x} = \vec{b}$ is an inconsistent system for $\vec{x}$



Then solution(s) to

$$A\vec{x} = \text{proj}_{\text{Im}(A)} \vec{b}$$

are called
least-squares solution(s)

because they minimize $\|A\vec{x} - \vec{b}\|^2$

<u>Example</u>: $L(\vec{x}) = C\vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \vec{b}$

$\text{Image}(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ = plane $-y_1 + 2y_2 + y_3 = 0$ (see page 3!)

$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ is not on this plane!

Using o.n. basis from page 3,

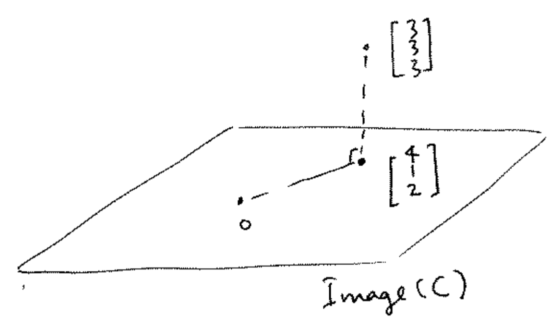
$$\text{proj}_V \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = (\vec{b} \cdot \vec{w}_1)\vec{w}_1 + (b \cdot \vec{w}_2)\vec{w}_2$$

$$= \left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right]$$

$x_1 = 2$, $x_2 = 1$; $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ ✓



There's a smarter way to do this:

$$A\vec{x} = \text{proj}_V \vec{b} \quad , \quad V = \text{image}(A)$$

$$\Leftrightarrow \vec{b} - A\vec{x} \in V^\perp$$

$\Leftrightarrow \vec{b} - A\vec{x}$ $\perp$ to a spanning set of V, e.g. the columns of A

$$\Leftrightarrow A^T(b - A\vec{x}) = 0$$

$$\Leftrightarrow \boxed{A^T A \vec{x} = A^T \vec{b}}$$

least squares soln to $A\vec{x} = \vec{b}$

no GS needed!

page 4 example cont'd:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \checkmark$$

On Friday, in the lab LCB 115, I will explain how this so-called "method of least squares" is related to the method of least squares in "linear regression", getting a "best" line to approximate a collection of points

(or you can look in §5.4)