Math 2270-1
Monday 17 Oct

Finish Friday's notes on §5.1. Then §5.2: <u>Finding orthonormal bases.</u>

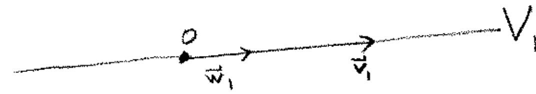
So how do you find an orthonormal basis for a subspace $V \subset \mathbb{R}^n$?

Start with any basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ ...

the process is inductive. It is called <u>Gram-Schmidt orthogonalization</u>

Let $V_1 = \operatorname{span}\{\vec{v}_1\}$

$\vec{w}_1 := \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$, then $V_1 = \operatorname{span}\{\vec{w}_1\}$

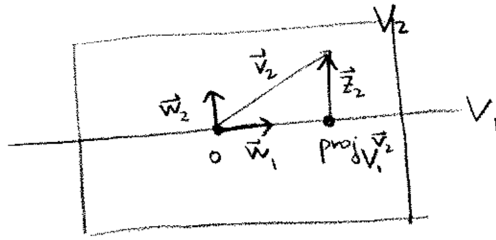


Let $V_2 = \operatorname{span}\{\vec{v}_1, \vec{v}_2\}$

let $\vec{z}_2 = \vec{v}_2 - \operatorname{proj}_{V_1}\vec{v}_2$

$\vec{w}_2 = \dfrac{\vec{z}_2}{\|\vec{z}_2\|}$

then $V_2 = \operatorname{span}\{\vec{w}_1, \vec{w}_2\}$

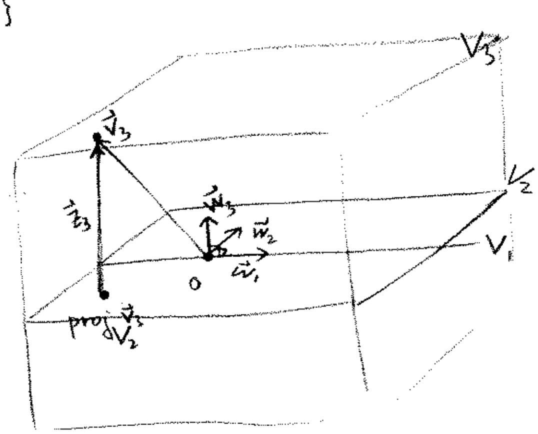


Let $V_3 = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\vec{z}_3 = \vec{v}_3 - \operatorname{proj}_{V_2}\vec{v}_3$

$\vec{w}_3 := \dfrac{\vec{z}_3}{\|\vec{z}_3\|}$

then $V_3 = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$



inductively,

$V_j := \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_j\}$

$\vec{z}_j := \vec{v}_j - \operatorname{proj}_{V_{j-1}}\vec{v}_j = \vec{v}_j - \left[(\vec{v}_j \cdot \vec{w}_1)\vec{w}_1 + (\vec{v}_j \cdot \vec{w}_2)\vec{w}_2 + \ldots + (\vec{v}_j \cdot \vec{w}_{j-1})\vec{w}_{j-1}\right]$

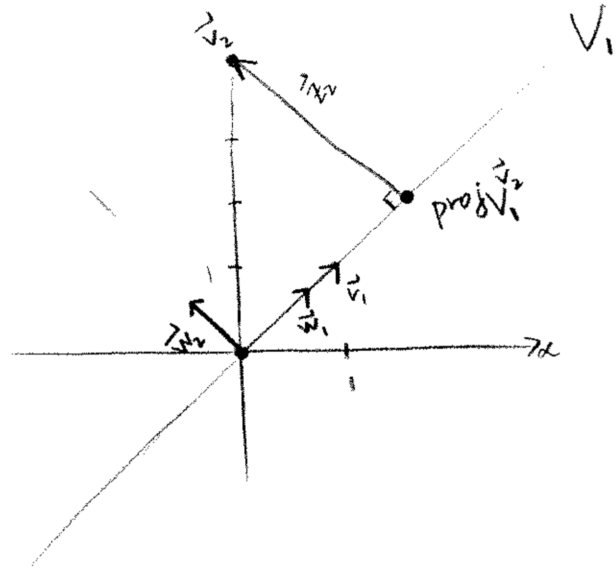
$\vec{w}_j := \dfrac{\vec{z}_j}{\|\vec{z}_j\|}$

then $V_j = \operatorname{span}\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_j\}$

$j = 2, 3, \ldots k$.

Examples

$$\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}0\\4\end{bmatrix}} \right\}$$



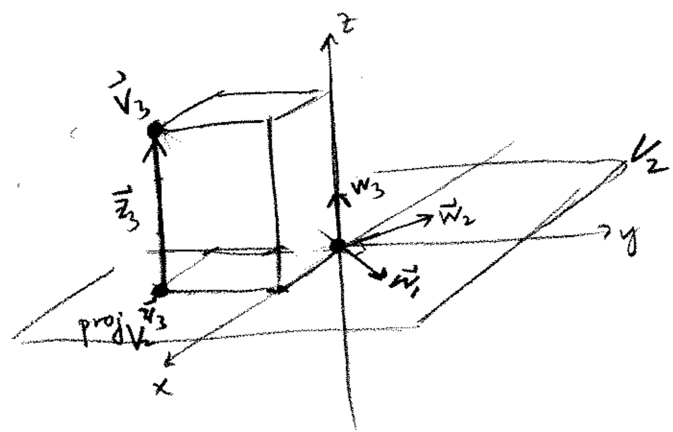
$$\vec{w}_1 := \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{aligned}
\vec{z}_2 &= \vec{v}_2 - \operatorname{proj}_{V_1}\vec{v}_2 \\
&= \vec{v}_2 - (\vec{v}_2\cdot\vec{w}_1)\vec{w}_1 \\
&= \begin{bmatrix}0\\4\end{bmatrix} - \frac{4}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\
&= \begin{bmatrix}-2\\2\end{bmatrix}
\end{aligned}$$

$$\vec{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}.$$

orthonormal basis $\left\{ \underset{\vec{w}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{w}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}} \right\} = \mathcal{O}$

$$\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}0\\4\\0\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}1\\-2\\3\end{bmatrix}} \right\}$$



proceeds as first example until step 3

$$\vec{w}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\begin{aligned}
\vec{z}_3 &= \vec{v}_3 - \operatorname{proj}_{V_2}\vec{v}_3 \\
&= \vec{v}_3 - (\vec{v}_3\cdot\vec{w}_1)\vec{w}_1 - (\vec{v}_3\cdot\vec{w}_2)\vec{w}_2 \\
&= \begin{bmatrix}1\\-2\\3\end{bmatrix} - \frac{(-1)(1)}{\sqrt{2}\,\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} - \left(\frac{-3}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} \\
&= \begin{bmatrix}1\\-2\\3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} \\
&= \begin{bmatrix}0\\0\\3\end{bmatrix}
\end{aligned}$$

$$\vec{w}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad !$$

orthonormal basis $\left\{ \underset{\vec{w}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{w}_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}}, \underset{\vec{w}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\} = \mathcal{O}$

Gram-Schmidt lets you get the orthonormal basis $\mathcal{O}$ from $\mathcal{B}$.
(Of course?) it is then easy to do the reverse, namely express $\mathcal{B}$ in terms of $\mathcal{O}$
--- BECAUSE FOR ORTHONORMAL BASES THE COORDINATES ARE EASY TO FIND!

$$\vec{v}_1 = (\vec{v}_1 \cdot \vec{w}_1)\vec{w}_1$$

$$\vec{v}_2 = (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 + (\vec{v}_2 \cdot \vec{w}_2)\vec{w}_2$$

$$\vec{v}_3 = (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 + (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2 + (\vec{v}_3 \cdot \vec{w}_3)\vec{w}_3$$

$$\vdots$$

$$\vec{v}_k = (\vec{v}_k \cdot \vec{w}_1)\vec{w}_1 + \cdots + (\vec{v}_k \cdot \vec{w}_k)\vec{w}_k$$

ij entry is $\vec{w}_i \cdot \vec{v}_j$

We usually write this in matrix form:

$$\left[\begin{array}{c|c|c|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{array}\right] = \underbrace{\left[\begin{array}{c|c|c|c} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{array}\right]}_{\text{columns are ortho-normal.}} \underbrace{\begin{bmatrix} \vec{w}_1\cdot\vec{v}_1 & \vec{w}_1\cdot\vec{v}_2 & \vec{w}_1\cdot\vec{v}_3 & \cdots & \vec{w}_1\cdot\vec{v}_k \\ 0 & \vec{w}_2\cdot\vec{v}_2 & \vec{w}_2\cdot\vec{v}_3 & \cdots & \\ 0 & 0 & \vec{w}_3\cdot\vec{v}_3 & \cdots & \\ 0 & 0 & 0 & \ddots & \\ & & & & \vec{w}_k\cdot\vec{v}_k \end{bmatrix}}_{k\times k \text{ upper } \Delta\text{'lar. matrix}}$$

notice the diagonal elts $\vec{w}_j \cdot \vec{v}_j = \|\vec{z}_j\| \neq 0$

Thus Any matrix A with linear ind. columns may be written as

$$A_{m\times k} = Q_{m\times k} R_{k\times k} \qquad \text{as above.}$$

What happens if the columns are not independent?

example

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0\\4\end{bmatrix}\right\} \qquad \mathcal{O}=\left\{\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\tfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right\}$$

$$\quad\ \vec{v}_1 \quad \vec{v}_2 \qquad\qquad\qquad \vec{w}_1 \qquad\quad \vec{w}_2$$

so $$\begin{bmatrix}1&0\\1&4\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\vec{w}_1\cdot\vec{v}_1&\vec{w}_1\cdot\vec{v}_2\\0&\vec{w}_2\cdot\vec{v}_2\end{bmatrix}$$

$$\begin{bmatrix}1&0\\1&4\end{bmatrix}=\underbrace{\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\sqrt{2}&2\sqrt{2}\\0&2\sqrt{2}\end{bmatrix}}_{R}$$

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\4\\0\end{bmatrix},\begin{bmatrix}1\\-2\\3\end{bmatrix}\right\} \qquad \mathcal{O}=\left\{\tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix},\tfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

$$\quad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$$

so $$\underbrace{\begin{bmatrix}1&0&1\\1&4&-2\\0&0&3\end{bmatrix}}_{A}=\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sqrt{2}&2\sqrt{2}&\overset{\vec{w}_2\cdot\vec{v}_3}{-1/\sqrt{2}}\\0&2\sqrt{2}&-3/\sqrt{2}\\0&0&3\end{bmatrix}$$

so $T(\vec{x})=A\vec{x}$ is a composition; first scale & shear (roughly speaking), then rotate.

the $A=QR$ decomposition has such geometric consequences as these, & more.