Math 2270-1
Wed Oct 12
EXTRA!

Maple Lab meeting will be Wed 10/19, not Mon 10/17, next week.

Partial fractions justified via linear algebra!
We shall do the simplest case; other cases may be treated analogously, but are technically harder.

Recall, If $q(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$ with all roots $r_1, r_2, \ldots r_n$ distinct ($r_i = r_j$ only if $i=j$)

and if degree $(p(x)) \le n-1$
Then we taught you in Calc that there are unique constants $A_1, A_2, \ldots A_n$ so that

$$\frac{p(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}.$$

WHY?

BECAUSE!

If you recombine the right side over the common denominator $q(x)$, and equate numerators you are seeing if you can find $A_1, A_2, \ldots A_n$ so that

$$p(x) = A_1 \underbrace{(x-r_2)(x-r_3)\cdots(x-r_n)}_{\substack{q_1(x) \\ \left(=\frac{q}{x-r_1}\right) \\ \text{for } x\ne r_1}} + A_2 \underbrace{(x-r_1)(x-r_3)\cdots(x-r_n)}_{\substack{q_2(x) \\ \left(=\frac{q}{x-r_2}\right) \\ \text{for } x\ne r_2}} + \cdots + A_n \underbrace{(x-r_1)(x-r_2)\cdots(x-r_{n-1})}_{\substack{q_n(x) \\ \left(=\frac{q}{x-r_n}\right) \\ \text{for } x\ne r_n}}$$

look in $\mathbb{P}_{n-1}$ (polys of degree $\le n-1$)
the question is, are $\{q_1, q_2, \ldots q_n\}$ a basis for $\mathbb{P}_{n-1}$? (which is n-dim'l).
We only need to check if they are linearly independent ($\Rightarrow$ span since # is correct)
But if

$$0 = A_1 q_1 + A_2 q_2 + \cdots + A_n q_n$$

at $x = r_j \Rightarrow 0 = A_1 q_1(r_j) + A_2 q_2(r_j) + \cdots + A_n q_n(r_j) = A_j q_j(r_j)$ (all others $=0$)

$$\Rightarrow A_j = 0, \quad j = 1, 2, \ldots n$$

$\blacksquare$

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• Try some chapter 3 T-F!

Chapter 5: Orthogonality in $\mathbb{R}^n$ and beyond!

recall

a) $\vec{v}, \vec{w} \in \mathbb{R}^n$, we say $\vec{v}$ and $\vec{w}$ are perpendicular, ($\vec{v} \perp \vec{w}$), or orthogonal iff $\vec{v} \cdot \vec{w} = 0$

b) $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$

c) $\vec{u} \in \mathbb{R}^n$ is a unit vector iff $\|\vec{u}\| = 1$

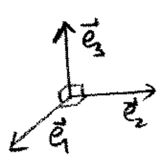
d) if $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, the unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$.

Bases for subspaces which consist of mutually orthogonal unit vectors, are really easy to work with, so have a special name:

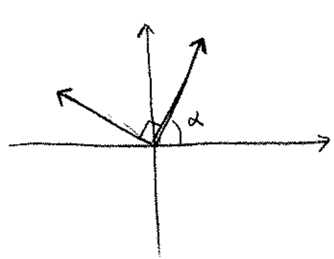
$$\boxed{\begin{array}{l} \text{The set } \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\} \text{ (in } \mathbb{R}^n) \text{ is called orthonormal iff } \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\ \text{[i.e. each } \vec{v}_i \text{ is unit length and } \vec{v}_i \perp \vec{v}_j \text{ for } i \neq j \end{array}}$$

examples we know

① standard basis $\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$ for $\mathbb{R}^n$



② rotated basis $\left\{ \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right\}$ for $\mathbb{R}^2$



Why orthonormal sets are good: (part 1)

Thm Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ be orthonormal. Then

(1) the set is linearly independent

(2) If $V = \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$, and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$

then $[v]_{\mathcal{B}} = \begin{bmatrix} \vec{v}\cdot\vec{v}_1 \\ \vec{v}\cdot\vec{v}_2 \\ \vdots \\ \vec{v}\cdot\vec{v}_k \end{bmatrix}$; in other words the linear combination coefficients for $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ can be computed with the dot product: $c_j = \vec{v}\cdot\vec{v}_j$.

proof:

(1) Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$.

take dot product with any $\vec{v}_j$: $\underbrace{\vec{v}_j \cdot (c_1\vec{v}_1 + \dots + c_k\vec{v}_k)}_{0+0+\dots+c_j+0+0} = \vec{v}_j\cdot\vec{0} = 0$

$0 + 0 + \dots + c_j + 0 + 0 = 0$

$c_j = 0$; $j = 1, 2, \dots k$.

(2) Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{v}$

take dot prod with $\vec{v}_j$: this time get

$c_j = \vec{v}\cdot\vec{v}_j$!

example: Consider the plane $V \subset \mathbb{R}^3$, $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - 5x_2 + x_3 = 0 \right\}$

$\mathcal{B} = \left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for V

find $\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}_{\mathcal{B}}$: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{5} \\ 12/\sqrt{6} \end{bmatrix}$! (how)

so $\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = \frac{5}{5}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \frac{12}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ ✓

<u>example</u>: $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \mathcal{B} = \left\{ \underset{\vec{u}}{\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{v}}{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}, \underset{\vec{w}}{\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}} \right\}$

find $[\vec{x}]_{\mathcal{B}}$:

$$\begin{bmatrix} \vec{x}\cdot\vec{u} \\ \vec{x}\cdot\vec{v} \\ \vec{x}\cdot\vec{w} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2+4+3) \\ \frac{1}{3}(1-4+6) \\ \frac{1}{3}(-2+2+6) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

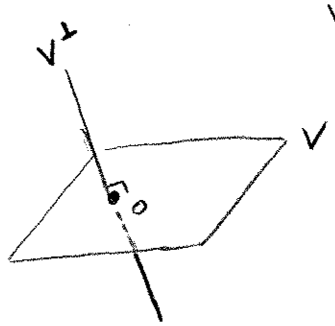
so $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ ✓

Why orthonormal sets are good: (part 2) They are good for projection problems

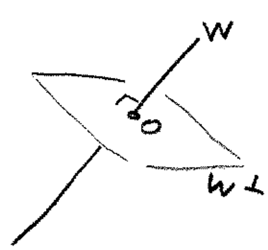
First, the notion of orthogonal complement:

<u>Def</u> Let $V \subset \mathbb{R}^n$ be a subspace. $V^\perp$, the orthogonal complement to V is defined by

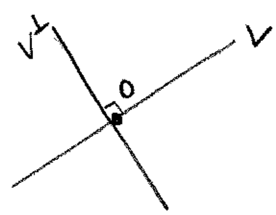
$$V^\perp := \{ \vec{x} \in \mathbb{R}^n : \vec{x}\cdot\vec{v} = 0 \ \forall \vec{v} \in V \}.$$



$\mathbb{R}^3$



$\mathbb{R}^3$



$\mathbb{R}^2$

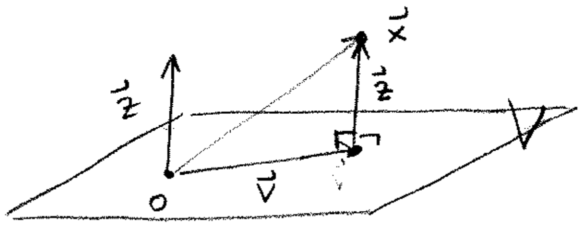
$V^\perp$ is a subspace of $\mathbb{R}^n$!

(a)

(b)

How do you think $\dim(V)$, $\dim(V^\perp)$, and "n" are related?

<u>Theorem</u> Let $V \subset \mathbb{R}^n$ a subspace. Let $\vec{x} \in \mathbb{R}^n$. Then there is a unique decomposition $\vec{x} = \vec{v} + \vec{z}$ where $\vec{v} \in V$ and $\vec{z} \in V^\perp$. $\vec{v}$ (hence $\vec{z} = \vec{x} - \vec{v}$) are easily computed using an orthonormal basis for V. $\vec{v}$ is called the projection of $\vec{x}$ onto V, $\mathrm{proj}_V \vec{x}$.



(we projected onto lines before)

proof, which tells us how to construct the projection!

Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ be an orthonormal basis for V [can always find one, see §5.2]

If $\vec{x} = \vec{v} + \vec{z}$ $\qquad \vec{v} \in V,\ \vec{z} \in V^\perp$

$\Rightarrow \vec{x} \cdot \vec{v}_j = (\vec{v} + \vec{z}) \cdot \vec{v}_j$

* $\Rightarrow \vec{x} \cdot \vec{v}_j = \vec{v} \cdot \vec{v}_j + \cancelto{0}{\vec{z} \cdot \vec{v}_j}$

But by page 2 discussion, the $\vec{v} \cdot \vec{v}_j$ are the linear combination coefficients of $\vec{v}$, so $\vec{v}$ <u>must</u> equal:

$$\vec{v} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \ldots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k = \sum_{j=1}^{k} (\vec{x} \cdot \vec{v}_j)\vec{v}_j$$

In this case $\vec{z}$ must equal $\vec{x} - \vec{v}$

$$\vec{z} = \vec{x} - \sum_{j=1}^{k} (\vec{x} \cdot \vec{v}_j)\vec{v}_j.$$

Is $\vec{z} \in V^\perp$?

true iff $\vec{z} \cdot \vec{v}_\ell = 0 \qquad \ell = 1, 2, \ldots k$

But $\vec{z} \cdot \vec{v}_\ell = \left(\vec{x} - \sum_{j=1}^{k} (\vec{x} \cdot \vec{v}_j)\vec{v}_j\right) \cdot \vec{v}_\ell$

$$= (\vec{x} \cdot \vec{v}_\ell) - \left[(\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \cdots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k\right] \cdot \vec{v}_\ell$$

$$= (\vec{x} \cdot \vec{v}_\ell) - \left[0 + 0 + \cdots + (x \cdot \vec{v}_\ell) \cdot 1 + 0 + \cdots 0\right]$$

$$= 0!$$

<u>example</u> (continued from page 2) $V = \{2x_1 - 5x_2 + x_3 = 0\}$

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix} \right\}$$

Let $\vec{x} = \begin{bmatrix}9\\-9\\-3\end{bmatrix}$. Find $\mathrm{proj}_V \vec{x}$

$$\begin{aligned}\mathrm{proj}_V \vec{x} &= (\vec{x}\cdot\vec{u})\vec{u} + (\vec{x}\cdot\vec{v})\vec{v} \\ &= \frac{1}{\sqrt{5}} 15 \frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix} + \frac{1}{\sqrt{6}} 6 \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix} \\ &= 3\begin{bmatrix}1\\0\\-2\end{bmatrix} + \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}5\\1\\-5\end{bmatrix}.\end{aligned}$$

so $\vec{x} = \vec{v} + \vec{z}$

$$\begin{aligned}\vec{z} &= \vec{x} - \vec{v} \\ &= \begin{bmatrix}9\\-9\\-3\end{bmatrix} - \begin{bmatrix}5\\1\\-5\end{bmatrix} = \begin{bmatrix}4\\-10\\2\end{bmatrix} \in V^\perp\end{aligned}$$

$\left[V^\perp = \mathrm{span}\left\{\begin{bmatrix}2\\-5\\1\end{bmatrix}\right\} \right]$ ✓

decomposition

$$\vec{x} = \vec{v} + \vec{z}$$

$$\begin{bmatrix}9\\-9\\-3\end{bmatrix} = \begin{bmatrix}5\\1\\-5\end{bmatrix} + \begin{bmatrix}4\\-10\\2\end{bmatrix}$$ ✓

to be continued...