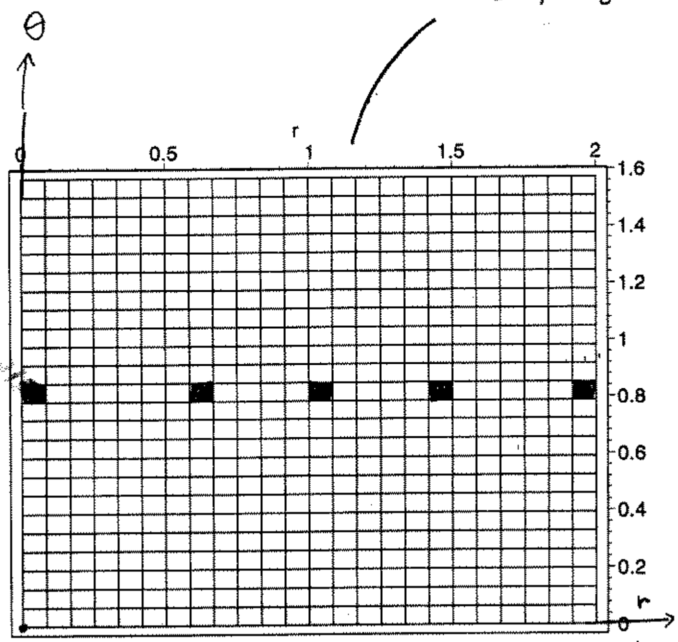
Math 2270-1
Wed 11/8 : EXTRA!!

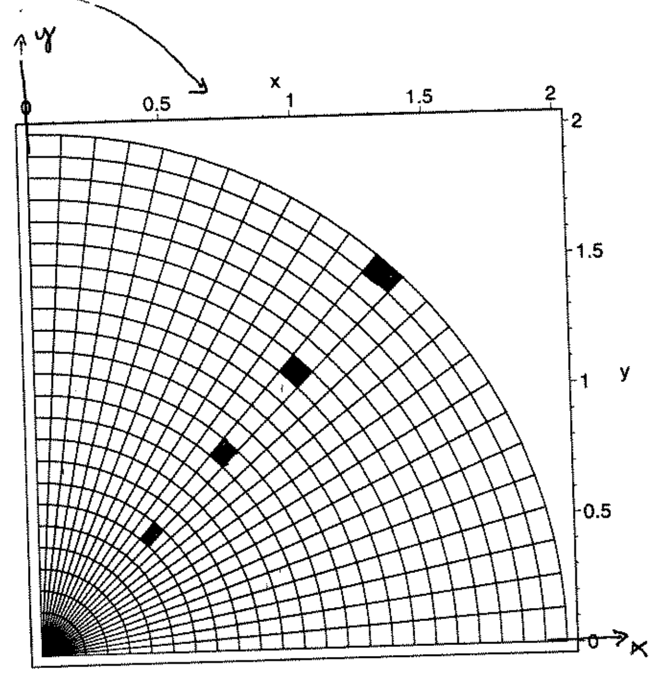
Geometry of determinants in multiple integration, & change of variables

Differentiable transformations $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ are well-approximated by affine approximations on small scales.

Here's a picture of what I mean, for the polar coordinate transformation:

$$F\left(\begin{bmatrix} r \\ \theta \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$





$0 \le r \le 2$
$0 \le \theta \le \pi/2$

In 2210 you computed $\Delta A = \left(\frac{r_1+r_2}{2}\right)\left(\frac{\theta_1+\theta_2}{2}\right)\Delta r \Delta\theta$

so $dA = r\, dr\, d\theta$

$\uparrow$ area expansion factor.
Is this a determinant of something?

so, for example,

$$\text{sector area} = \iint 1\, dA = \int_0^{\pi/2}\int_0^2 1 \cdot r\, dr\, d\theta = \int_0^{\pi/2} \frac{r^2}{2}\Big]_0^2 d\theta = 2 \cdot \pi/2 = \boxed{\pi}$$

$$\left(= \tfrac{1}{4}(\pi \cdot 2^2) \checkmark \right)$$

Determinants in multiple integrals:

recall (?)

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ (not necessarily linear)

is differentiable at $\vec{x}_0$

the $m \times n$ derivative matrix, ij entry is $\frac{\partial F_i}{\partial x_j}(\vec{x}_0)$

iff $\vec{F}(\vec{x}_0+\vec{h}) = \vec{F}(\vec{x}_0) + F'(\vec{x}_0)\vec{h} + \vec{\varepsilon}(\vec{h})$

where $\lim_{\vec{h}\to\vec{0}} \frac{\|\varepsilon(\vec{h})\|}{\|\vec{h}\|} = 0$

if you don't recall, or even if you do, let's remember how this affine approximation formula

$$F(x_0+h) \approx F(x_0) + F'(x_0)h$$

arises:

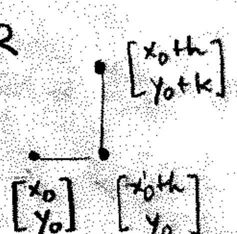
$f: \mathbb{R} \to \mathbb{R}$ $\quad \frac{f(x_0+h)-f(x_0)}{h} = f'(c)$ Mean value thm, c btwn x_0 & x_0+h

so $f(x_0+h) = f(x_0) + h f'(c)$

$f(x_0+h) \approx f(x_0) + h f'(x_0)$ [if f' is cont]

$f: \mathbb{R} \to \mathbb{R}^n$ $\quad \vec{f}(x_0+h) \approx \vec{f}(x_0) + h\vec{f}'(x_0)$ [argue for each component fcn].

$f: \mathbb{R}^2 \to \mathbb{R}$



$\mathbb{R}^2$ (domain)

$$f\begin{bmatrix} x_0+h \\ y_0+k \end{bmatrix} - f\begin{bmatrix} x_0+h \\ y_0 \end{bmatrix} = \frac{\partial f}{\partial y}\left(\begin{bmatrix} x_0+h \\ c \end{bmatrix}\right) k$$

$$+ \quad f\begin{bmatrix} x_0+h \\ y_0 \end{bmatrix} - f\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} d \\ y_0 \end{bmatrix} h$$

$$f\begin{bmatrix} x_0+h \\ y_0+k \end{bmatrix} - f\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \approx h\frac{\partial f}{\partial x}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + k\frac{\partial f}{\partial y}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$f\begin{pmatrix} x_0+h \\ y_0+k \end{pmatrix} \approx f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]\begin{bmatrix} h \\ k \end{bmatrix}$$

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(\vec{x}_0+\vec{h}) \approx f(\vec{x}_0) + \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n}\right]\begin{bmatrix} h_1 \\ h_2 \\ h_n \end{bmatrix}$

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \vec{F}(\vec{x}_0+\vec{h}) \approx \vec{F}(\vec{x}_0) + \left[\begin{array}{c|c|c|c} \frac{\partial \vec{f}}{\partial x_1} & \frac{\partial \vec{f}}{\partial x_2} & & \frac{\partial \vec{f}}{\partial x_n} \end{array}\right]\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$

Math 2210-1
~~Wed Nov. 17~~

~~Practice exam review session tomorrow (Thurs.) LCB 121, 9:40-10:30~~

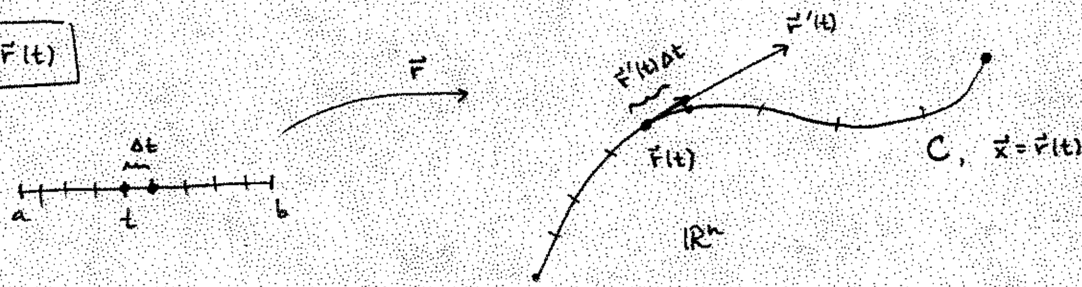
old 2210 notes!

~~HW for Wed 11/24~~
~~§ 14.9 3, 7, 8, 13, 14, 22~~

§ 14.9 Change of variables in multiple integration

this discussion fits into a bigger picture discussion of integrals for parametric objects:

$\boxed{\bullet \text{ curves } \ \vec{r}(t)}$



$$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{r}\,'(t)\Delta t \qquad \text{(ignoring "errors")}$$

$$\Delta\vec{r} \approx \vec{r}\,'(t)\Delta t$$

$$ds = |\vec{r}\,'(t)|\,dt \qquad \text{element of arclength}$$

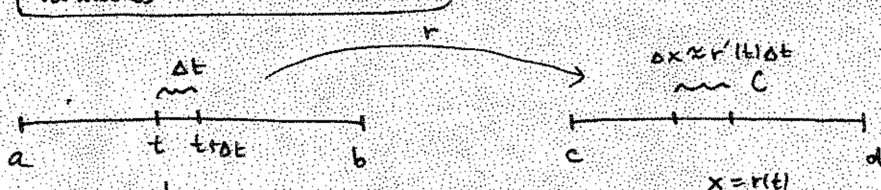
(Section 12.2) length $\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$

length expansion factor

(15.2) More generally, $\boxed{\int_C f(\vec{x})\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt}$

e.g. f could be scalar density fcn (mass/length) and you could be computing total mass.

Special case is change of variables in 1-var. Calc:

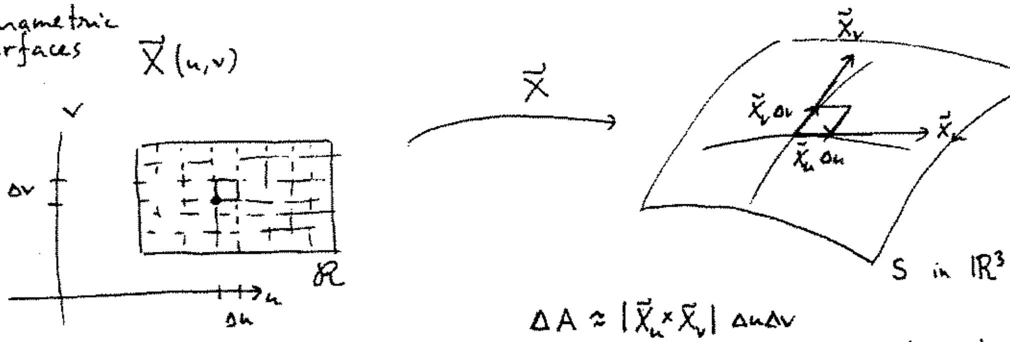


picture when $r'(t) > 0$ on $[a,b]$,

$$\int_C f(x)\,ds = \int_c^d f(x)\,dx = \int_a^b f(r(t))\,r'(t)\,dt$$

i.e. you've substituted
$x = r(t)$
$dx = r'(t)\,dt$

• parametric surfaces $\vec{X}(u,v)$



$\Delta A \approx |\vec{X}_u \times \vec{X}_v|\,\Delta u \Delta v$

$dA = |\vec{X}_u \times \vec{X}_v|\,du\,dv$ area element

(based on $\vec{X}(u+\Delta u, v+\Delta v) \approx \vec{X}(u,v) + \underbrace{\left[\vec{X}_u \middle| \vec{X}_v\right]\begin{bmatrix}\Delta u \\ \Delta v\end{bmatrix}}_{= \vec{X}_u \Delta u + \vec{X}_v \Delta v}$)

§14.8 so, $$\boxed{A = \iint_R |\vec{X}_u \times \vec{X}_v|\,du\,dv}$$

more generally,

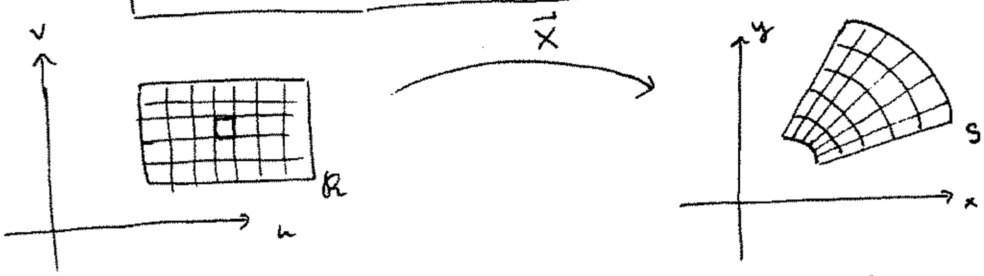
§15.5 $$\boxed{\iint_S f(\vec{x})\,dA = \iint_R f(\vec{X}(u,v))\,|\vec{X}_u \times \vec{X}_v|\,du\,dv}$$

2270 this also equals $\sqrt{|A^T A|}$

$$= \sqrt{\det\left(\begin{bmatrix} \vec{x}_u^T \\ \vec{x}_v^T \end{bmatrix}\left[\vec{X}_u \middle| \vec{X}_v\right]\right)}$$

check!

Special case is change of variables in 2-var integrals:



think of $\vec{X}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ 0 \end{bmatrix}$

$$|\vec{X}_u \times \vec{X}_v| = \text{abs}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix}$$

$$= \text{abs}\left(\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}\right)$$

$$= \text{abs}\Bigg(\underbrace{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}}\Bigg)$$

determinant of deriv matrix for $\begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$

called Jacobian det,

write $\dfrac{\partial(x,y)}{\partial(u,v)}$

so, $$\boxed{\iint_S f(x,y)\,dA = \iint_R f(\vec{X}(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

2270

know without cross product!

Examples

(1) $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{X}\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

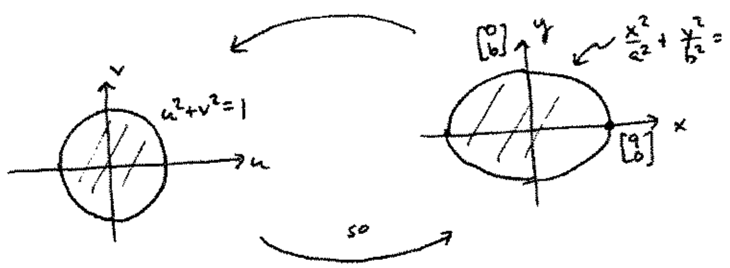
derivative matrix is :

determinant is :

deduce. $\displaystyle \iint_{\substack{x-y \\ \text{region}}} f(x,y)\,dA = \iint_{\substack{r-\theta \\ \text{region}}} f(r\cos\theta, r\sin\theta)\,\overset{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}{\overset{\downarrow}{r}}\,dr\,d\theta$

$u = \frac{x}{a},\ v = \frac{y}{b}$

(2) Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



2270:

$\vec{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

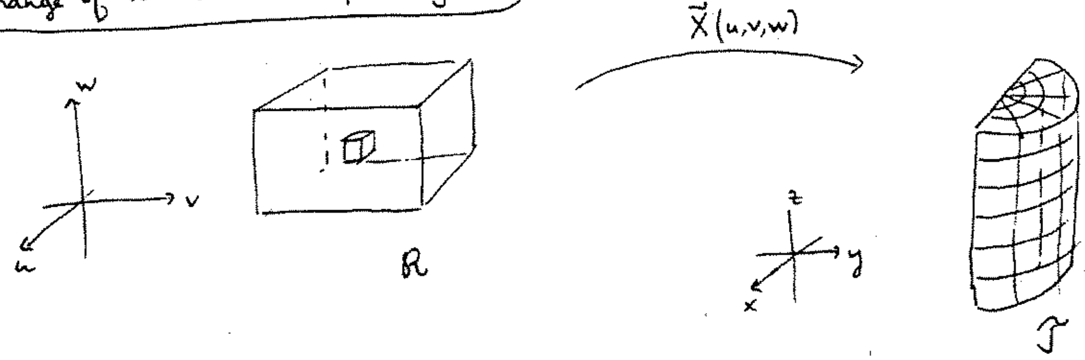
det = ab

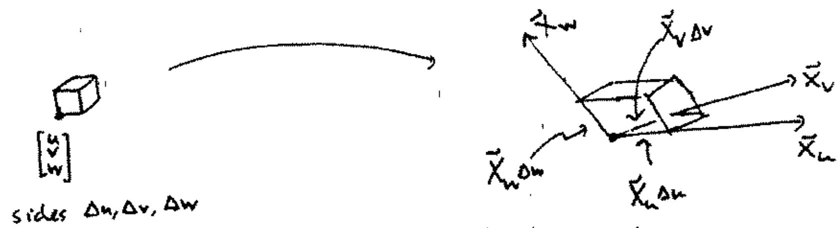
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} au \\ bv \end{bmatrix}$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; det = ab

so $\displaystyle \iint dA = \iint ab\,du\,dv = \boxed{ab\,\pi}$

Change of variables in triple integrals





$$\Delta V \approx \text{abs}\Big(\underbrace{\left| \vec{X}_u \,\vdots\, \vec{X}_v \,\vdots\, \vec{X}_w \right|}_{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|} \Big)\, \Delta u \Delta v \Delta w$$

based on $\vec{X}(u+\Delta u, v+\Delta v, w+\Delta w) \approx \vec{X}(u,v,w) + \left[\vec{X}_u \,\vdots\, \vec{X}_v \,\vdots\, \vec{X}_w \right] \begin{bmatrix}\Delta u\\ \Delta v\\ \Delta w\end{bmatrix}.$

$$\iiint_{\mathcal{T}} f \, dV = \iiint_{\mathcal{R}} f(\vec{X}(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

<u>check</u>

Cylindrical Coords

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{X}\begin{bmatrix}r\\ \theta\\ z\end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z\end{bmatrix} \qquad \text{deriv mat } \vec{X}' = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

det = r ✓

Spherical Coords.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{X}\begin{bmatrix}\rho\\ \phi\\ \theta\end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi\end{bmatrix} \qquad \left| \det(X') \right| = \rho^2 \sin\phi \;!$$

2270 check!