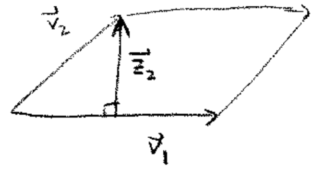
Math 2270-1
Tues 11/8

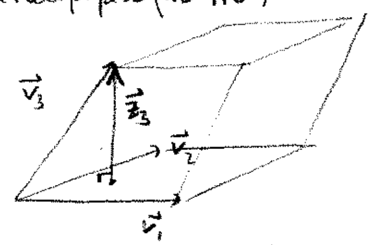
Geometry of determinants: Volumes & orientation
related to Gram Schmidt ideas!

parallelogram: (in $\mathbb{R}^n$)



Area $= \|\vec{v}_1\|\|\vec{z}_2\|$ (base·ht)

parallelepiped (in $\mathbb{R}^n$)



Vol = (Area of base) ht
$= \|\vec{v}_1\|\|\vec{z}_2\|\|\vec{z}_3\|$

in higher dimensions,

Vol of parallelepiped spanned by
$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is $\|\vec{v}_1\|\|\vec{z}_2\|\cdots\|\vec{z}_k\|$

$\vec{w}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{z}_2 = \vec{v}_2 - \mathrm{proj}_{V_1}\vec{v}_2$
$= \vec{v}_2 - (\vec{v}_2\cdot\vec{w}_1)\vec{w}_1$

$\vec{w}_2 = \dfrac{\vec{z}_2}{\|z_2\|}$

$\vec{z}_3 = \vec{v}_3 - \mathrm{proj}_{V_2}\vec{v}_3$
$= \vec{v}_3 - (\vec{v}_3\cdot\vec{w}_1)\vec{w}_1 - (\vec{v}_3\cdot\vec{w}_2)\vec{w}_2$

$\vec{w}_3 = \dfrac{\vec{z}_3}{\|z_3\|}$ etc.

relates to $A = QR$ decomposition:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ 0 & r_{22} & & \\ \vdots & \vdots & & \\ 0 & 0 & & r_{kk} \end{bmatrix}$$

$\vec{z}_j = \vec{v}_j - \mathrm{proj}_{V_{j-1}}\vec{v}_j$
$\vec{w}_j = \dfrac{\vec{z}_j}{\|\vec{z}_j\|}$

Look at R more closely: ["recall"]

$A = QR$
$Q^T A = Q^T Q R = IR$
$Q^T A = R$

$$R = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_k \end{bmatrix}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = \begin{bmatrix} r_{11} & & \text{stuff} \\ & r_{22} & \\ O & & r_{kk} \end{bmatrix}$$

$r_{11} = \vec{w}_1\cdot\vec{v}_1 = \|\vec{v}_1\|$
$r_{22} = \vec{w}_2\cdot\vec{v}_2 = \vec{w}_2\cdot\vec{z}_2 = \dfrac{\vec{z}_2}{\|\vec{z}_2\|}\cdot\vec{z}_2 = \|\vec{z}_2\|$

$j \geq 2$: $r_{jj} = \vec{w}_j\cdot\vec{v}_j = \vec{w}_j\cdot\vec{z}_j$
$= \dfrac{\vec{z}_j}{\|\vec{z}_j\|}\cdot\vec{z}_j = \|\vec{z}_j\|$

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} \|\vec{v}_1\| & r_{12} & \cdots & r_{1k} \\ & \|\vec{z}_2\| & & \vdots \\ 0 & & \ddots & \\ & & & \|\vec{z}_k\| \end{bmatrix}$$

$$\boxed{\text{Vol} = \det(R)\ !}$$

and we don't need Gram-Schmidt to compute this

$$A = QR$$
$$A^T A = (QR)^T QR$$
$$= R^T Q^T Q R$$
$$A^T A = R^T R$$

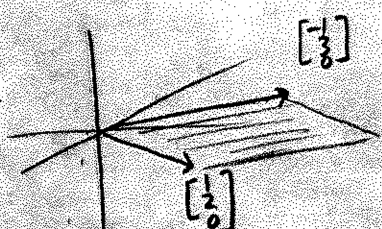
take det:

$$\det(A^T A) = \det(R^T R) = (\det R^T)(\det R)$$
$$\det(A^T A) = (\det R)^2$$

$$\boxed{\text{Vol} = \sqrt{\det(A^T A)}}$$

$$\boxed{\text{Special case : if } A_{n\times n} \text{ (n-dim'l box in } \mathbb{R}^n) \text{ then Vol} = |\det(A)|}$$

<u>examples</u>



$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}$$

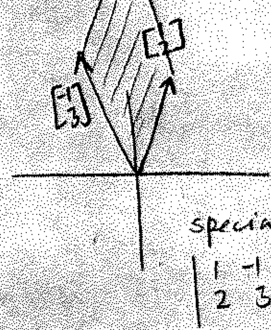
$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$$

$$\det(A^T A) = 25$$

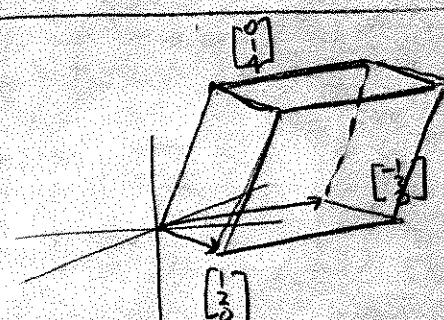
$$\boxed{\text{area} = 5}$$

$= \|\vec{u} \times \vec{v}\|$, by the way.



special case

$$\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 = \text{area.}$$



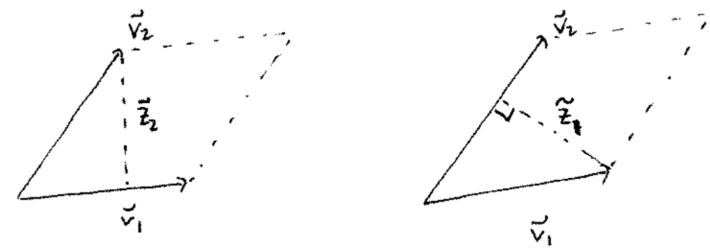
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$|A| = 12 + 8 = 20$$

$$\boxed{\text{Vol} = 20} \quad (= 5 \cdot 4) \checkmark$$

Can you explain why we get the same value for area or volume, no matter the order we list the spanning vectors?

e.g.



$$A = \|\vec{v}_1\|\,\|\vec{z}_2\| \overset{?}{=} \|\vec{v}_2\|\,\|\tilde{z}_1\|$$

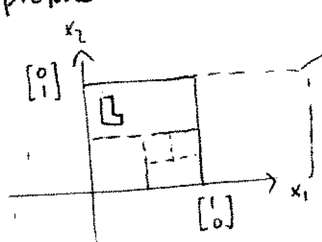
$$= \sqrt{\det\left(\begin{bmatrix} \vec{v}_1^{\,T} \\ \vec{v}_2^{\,T} \end{bmatrix}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\right)} \qquad = \sqrt{\det\left(\begin{bmatrix} \vec{v}_2^{\,T} \\ \vec{v}_1^{\,T} \end{bmatrix}\begin{bmatrix} \vec{v}_2 & \vec{v}_1 \end{bmatrix}\right)}$$

answer:

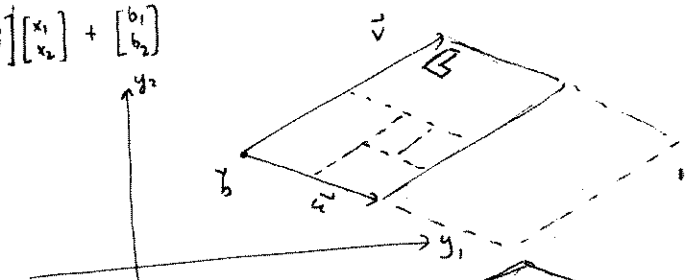
|Determinant| as area/volume expansion factor:
(Remember poor Bob).

If $f(\vec{x}) = A\vec{x}$, $f: \mathbb{R}^n \to \mathbb{R}^n$
Then $|\det A|$ is the ratio of $\dfrac{\text{area (or volume) of transformed Bob}}{\text{area (or volume) of original Bob}}$

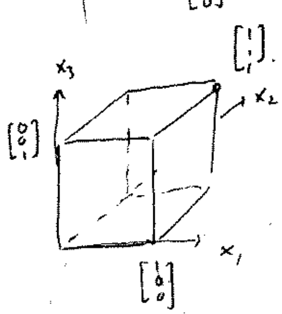
$n=2$ picture



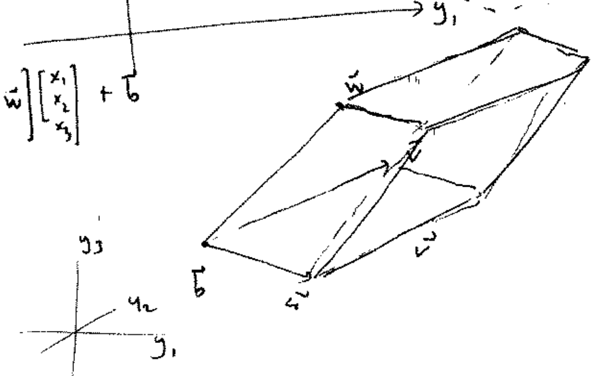
$$f(\vec{x}) = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



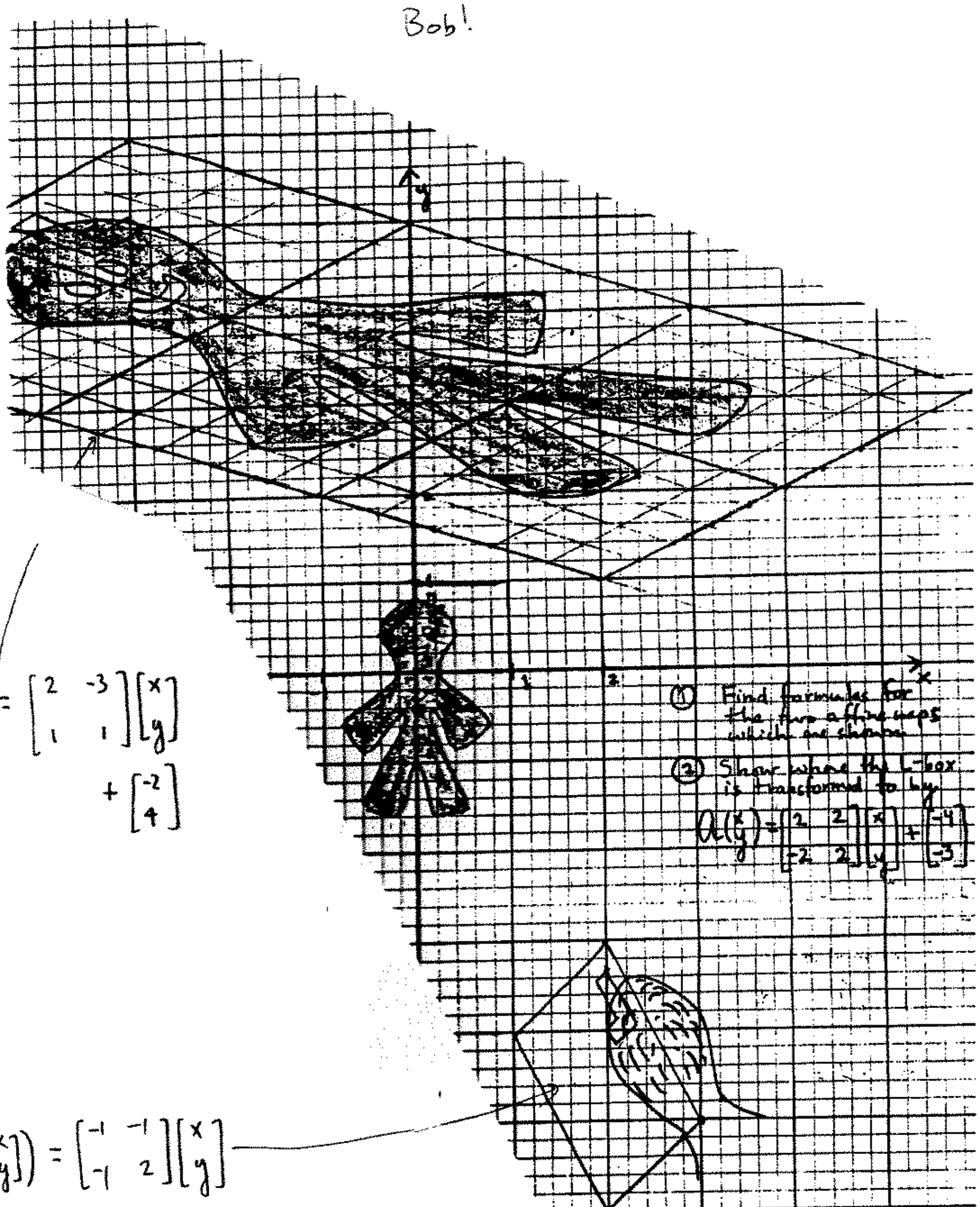
$n=3$



$$f(\vec{x}) = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \vec{b}$$



Bob!



$$a\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$a\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

The meaning of whether $\det A > 0$ or $\det A < 0$, in $n \times n$ case:

$A = QR$ $\quad$ Q is $(n \times n)$ orthogonal (provided A nonsingular)

$|A| = |Q||R|$

$|Q| = \pm 1$; the sign of $\det Q$ (equivalently, of $\det(A)$) determines the "orientation" of the columns of A

($\det Q = +1$ "positively oriented" ("right-handed")

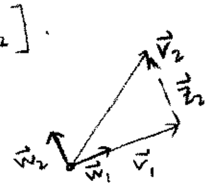
($\det Q = -1$ "negatively oriented" ("left-handed")

"negatively oriented": any n!

"left-handed": only really makes sense in $\mathbb{R}^2$, $\mathbb{R}^3$, because we don't have enough fingers for hands in $\mathbb{R}^n$, $n > 3$:)

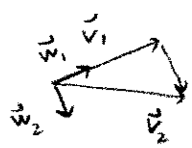
e.g. $n = 2$:

$A = [\vec{v}_1 | \vec{v}_2]$.



$$Q = \begin{bmatrix} \cos\alpha & \cos(\alpha + \pi/2) \\ \sin\alpha & \sin(\alpha + \pi/2) \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$\det Q = +1$ so $\det A > 0$



$$Q = \begin{bmatrix} \cos\alpha & \cos(\alpha - \pi/2) \\ \sin\alpha & \sin(\alpha - \pi/2) \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

$\det Q = -1$ so $\det A < 0$

similar in $n = 3$, $n > 3$.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det = +1$ $\rightarrow$ we'll show later that for $n = 3$, when $\det Q = +1$, Q really is a rotation about some axis

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det = -1$ and (for $n = 3$), if $\det Q = -1$, it's a reflection composition of a rotation & reflection (not necessarily a pure reflection)

... so for $f(\vec{x}) = A\vec{x} + \vec{b}$,

$\det A > 0$ means orientation is preserved

$\det A < 0$ means orientation is reversed!

See Bob!