Math 2270-1
Monday 11/7

Determinants! : Finish algebra today; geometry tomorrow

- $|A| = \underbrace{\sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|}_{\text{col}_j} = \underbrace{\sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|}_{\text{row}_i}$

- If $A$ is upper or lower $\Delta$'ular, $|A| = a_{11}a_{22}\cdots a_{nn}$
- Determinants and row operations (or column ops!)
  - (1) switching rows changes sign of det)
  - (2) multiplying a row by $c$ mults det by $c$
    (so factoring a $c$ out of a row, factors a $c$ out of det)
  - (3) replace $\text{row}_k$ by $\text{row}_k + c\,\text{row}_i$ $(i \neq k)$ leaves det unchanged.

example

$$\begin{vmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 3 & 6 & 9 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 6 & 9 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -R_1+R_3 \end{matrix}$$

$$= 6\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} \begin{matrix} \\ \\ -2R_2+R_3 \end{matrix}$$

$$= 18\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 18\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 18$$

example

$$\begin{vmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 5 & 3 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 5 & 3 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -5R_1+R_3 \end{matrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} \\ \\ -3R_2+R_3 \end{matrix}$$

$$= 2 \cdot 0 = 0.$$

Theorem!

$$\boxed{\det A \neq 0 \text{ iff } \text{rref}(A) = I \text{ iff } A^{-1} \text{ exists}}$$

because $\det A = \underbrace{k_1 k_2 \cdots k_\ell}_{\substack{\neq 0 \\ \text{(effects of row ops)}}} \underbrace{\det(\text{rref}(A))}_{\neq 0 \text{ iff } \text{rref}(A) = I}$ ∎

Corollary of above argument:

**Theorem** $\det(AB) = (\det A)(\det B)$

pf

doing rowops

$$
\begin{aligned}
|A| &= k_1 |A_1| \\
&= k_1 k_2 |A_2| \\
&\;\vdots \\
&= k_1 k_2 \cdots k_\ell \det(\mathrm{rref}(A))
\end{aligned}
$$

do same rowops of $AB$: [is same as doing rowops on $A$ & then multiplying by $B$]

$$
\begin{aligned}
|AB| &= k_1 |A_1 B| \\
&= k_1 k_2 |A_2 B| \\
&\;\vdots \\
&= k_1 k_2 \cdots k_\ell \det(\mathrm{rref}(A)\, B)
\end{aligned}
$$

Case 1: $\mathrm{rref}(A) = I$

In this case we see $|A| = k_1 k_2 \cdots k_\ell$ from

But then we see from that $|AB| = |A| \det(IB) = |A||B|$.

Case 2: $\mathrm{rref}(A) \neq I$. In this case both $|A| = 0$ and $|AB| = 0$, so $|AB| = |A||B|$

So how are $|A|$ and $|A^{-1}|$ related?

So if $A$ and $B$ are similar, $B = S^{-1}AS$, how are $|B|$ and $|A|$ related?

Def: Let $T: V \to V$ be linear, $\dim V = n$.

Then $\det(T) := \det B$, where $B = [T]_{\mathcal{B}}$, $\mathcal{B} = \{f_1, f_2, \ldots f_n\}$ any basis

<u>Theorem</u> $\det(A^T) = \det(A)$

<u>proof 1</u>: use induction; $n=1$ case clear. For inductive step expand $A$ down $1^{st}$ col, and $A^T$ across $1^{st}$ row. Notice the minor matrices $(A_{i,1})^T = (A^T)_{1,i}$

(see text, page 266)

<u>proof 2</u>: row operations on $A$ correspond to col ops on $A^T$, and have exactly the same effect on det, so

$$\det(A) = \underset{\neq 0 \text{ (row ops)}}{(k_1 k_2 \cdots k_\ell)} \det(\text{rref}(A)) = \begin{cases} k_1 k_2 \cdots k_\ell & \text{if } \text{rref}(A) = I \\ 0 & \text{if } \text{rref}(A) \neq I \end{cases}$$

$$\det(A^T) = \underset{\text{(col ops)}}{(k_1 \cdots k_\ell)} \det(\text{rcef}(A^T)) = \begin{cases} k_1 k_2 \cdots k_\ell & \text{if } \text{rref}(A) = I \\ 0 & \text{if } \text{rref}(A) \neq I. \end{cases}$$

example page 1

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 3 & 6 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 6 \\ 4 & 1 & 9 \end{bmatrix}$$

$$|A| = 2\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 6 & 9 \end{vmatrix} \qquad |A^T| = 2\begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 6 \\ 2 & 1 & 9 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \qquad = 6\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\vdots \qquad \vdots$$

$$= 18\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad = 18\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

So, if $Q$ is an orthogonal matrix (so $Q^{-1} = Q^T$), what must $|Q|$ equal?

<u>only examples</u>, $n=2$

$$[Rot_\alpha] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad [Refl_{\alpha/2}] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

And, we kind of did this already ...

Recall cofactor matrix, for $A$:

$$[c_{ij}] = (-1)^{i+j} |A_{i,j}|$$

So $$\operatorname{col}_j(\operatorname{cof}(A)) \cdot \operatorname{col}_j(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{i,j}| = |A|$$

whereas, if $k \neq j$,

$$\operatorname{col}_j(\operatorname{cof}(A)) \cdot \operatorname{col}_k(A) = \left| \quad \underset{\substack{\uparrow \\ j^{th}\text{ col} \\ \text{expand here}}}{\operatorname{col}_k(A)} \quad \cdots \quad \underset{\substack{\uparrow \\ \operatorname{col}_k(A)}}{\vdots} \quad \right| = 0$$

(replace $\operatorname{col}_j(A)$ with $\operatorname{col}_k(A)$)

Hence, for $\operatorname{Adj}(A) := \operatorname{cof}(A)^T$
(adjoint),

$$\left[\operatorname{Adj}(A)\right]\left[A\right] = \det(A)\, I$$

so if $A^{-1}$ exists, it equals $\frac{1}{\det A}\operatorname{Adj}(A) = \frac{1}{\det A}\left[\operatorname{cof}(A)\right]^T$.

did examples on 1st det day.

example
$n=2$: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \operatorname{cof}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\quad \operatorname{cof}(A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\frac{1}{\det A}\left[\operatorname{cof}(A)\right]^T = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \checkmark$$

<u>Cramer's rule</u> If $A$ is invertible, and $A\vec{x} = \vec{b}$ (so $\vec{x} = A^{-1}\vec{b}$)

Then
$$\begin{aligned} x_j &= \operatorname{row}_j(A^{-1}) \cdot \vec{b} \\ &= \frac{1}{\det(A)} \operatorname{row}_j(\operatorname{Adj}(A)) \cdot \vec{b} \\ &= \frac{1}{\det(A)} \operatorname{col}_j(\operatorname{cof}(A)) \cdot \vec{b} \\ &= \frac{\left| \operatorname{col}_1(A) \cdots \vec{b} \cdots \operatorname{col}_n(A) \right|}{|A|} \end{aligned}$$

<u>Example</u>: Find $x_1, x_2$ so that

$$\begin{bmatrix} 7 & 2 \\ 13 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Cramer's rule

- Adjoint formula for $A^{-1}$