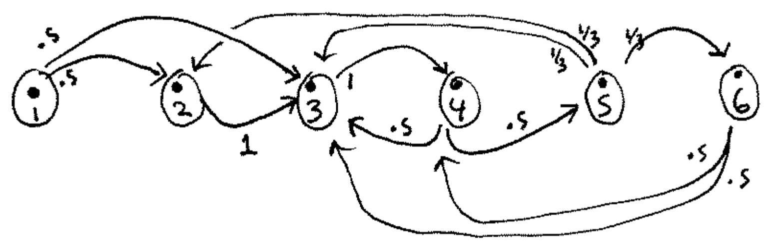
Math 2270-1
November 30, 2005
How Google works?

Link chart: for 6 sites on a certain topic




Discrete dynamical voting game: site j's current votes are sent to the sites it links to (in equal portions)

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> Digits:=4;
```

$$Digits := 4$$

```
> A:=matrix(6,6,[0,  0,0,0,0,0,
                 .5,0,0,0,1/3.,0,
                 .5,1,0,.5,1/3.,.5,
                 0,  0,1,0,0,.5,
                 0,  0,0,.5,0,0,
                 0,0,0,0,1/3.,0]);
```

regular transition matrix!

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.3333 & 0 \\ 0.5 & 1 & 0 & 0.5 & 0.3333 & 0.5 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3333 & 0 \end{bmatrix}$$

which site is the most valuable? (least?)

```
> evalm(A^100);
```

$$\begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. \\ 0.06037 & 0.06037 & 0.06038 & 0.06036 & 0.06037 & 0.06037 \\ 0.3321 & 0.3321 & 0.3322 & 0.3321 & 0.3321 & 0.3321 \\ 0.3622 & 0.3622 & 0.3622 & 0.3622 & 0.3622 & 0.3622 \\ 0.1812 & 0.1812 & 0.1813 & 0.1811 & 0.1812 & 0.1812 \\ 0.06037 & 0.06037 & 0.06038 & 0.06036 & 0.06037 & 0.06037 \end{bmatrix}$$

```
>
>
```

Math 2270-1
Wed 11/30

$\boxed{\text{MEET IN LCB 115 ON FRIDAY}}$

Example : (to motivate Chapter 8)

① Identify the curve $2x^2+2y^2+5xy=1$.
Can you graph it?

② Does the function $f(x,y) = 2x^2+2y^2+5xy$
have a local max or min at $\begin{bmatrix}0\\0\end{bmatrix}$?
(Note, $[f'] = [f_x, f_y] = [0,0]$ there,
so $\begin{bmatrix}0\\0\end{bmatrix}$ is a critical point.)

Ans : $2x^2+2y^2+5xy = [x, y]\begin{bmatrix}2 & 5/2\\ 5/2 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Check:

in general, if $\vec{v}, \vec{w}$ are column vectors, $A_{n\times n}$ then
$\vec{v}^T A\vec{w} = \sum_{i,j} v_i a_{ij} w_j$

pf: $\| \quad \sum_i v_i \underbrace{\text{entry}_i(A\vec{w})}_{\sum_j a_{ij}w_j}$

$|A-\lambda I|$

$= \begin{vmatrix}2-\lambda & 5/2\\ 5/2 & 2-\lambda\end{vmatrix}$

$= \lambda^2 - 4\lambda + (4-\frac{25}{4})$

$= \lambda^2 - 4\lambda - \frac{9}{4}$

$= (\lambda - \frac{9}{2})(\lambda+\frac{1}{2})$

$\lambda = 9/2$ :

$$\left[\begin{array}{cc|c} -5/2 & 5/2 & 0\\ 5/2 & -5/2 & 0\\ \hline 1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$.

$\lambda = -1/2$

$$\left[\begin{array}{cc|c} 5/2 & 5/2 & 0\\ 5/2 & 5/2 & 0\\ \hline 1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$\vec{v} = \begin{bmatrix}-1\\1\end{bmatrix}$.

is it an accident that the eigenspaces are orthogonal?

suggests a change of variables

$\mathcal{B} = \left\{ \begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} \right\}$

$[\vec{v}]_{\mathcal{B}} := \begin{bmatrix}x'\\y'\end{bmatrix}$.

$\begin{bmatrix}x\\y\end{bmatrix} = S\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}x'\\y'\end{bmatrix}$

& $\begin{bmatrix}x'\\y'\end{bmatrix} = S^T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$$2x^2+2y^2-5xy = [x \;\; y]\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}^T A \vec{x} = (\vec{x}')^T S^T A S \vec{x}' = \vec{x}'^T D \vec{x}'$$

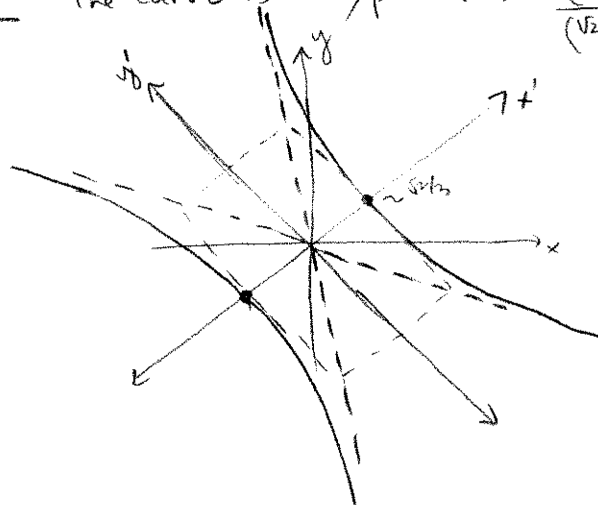
$$\vec{x} = S\vec{x}'$$

$$= [x' \;\; y']\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 5/2 \\ 5/2 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

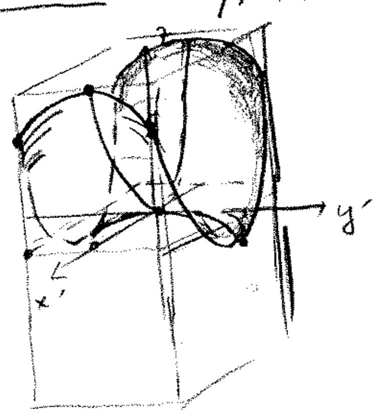
$$= [x' \;\; y']\begin{bmatrix} 9/2 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \tfrac{9}{2}(x')^2 - \tfrac{1}{2}(y')^2$$

ans to 1 the curve is a hyperbola! $\dfrac{(x')^2}{(\sqrt{2/3})^2} - \dfrac{(y')^2}{(\sqrt{2})^2} = 1$

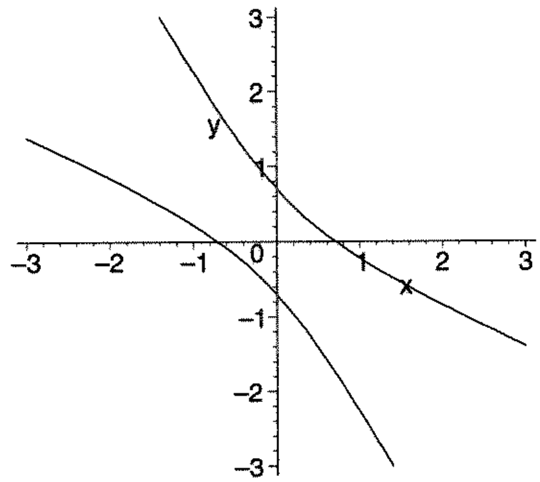



ans to 2 sorry, it's a saddle ; $g(x',y') = \tfrac{9}{2}(x')^2 - \tfrac{1}{2}(y')^2$

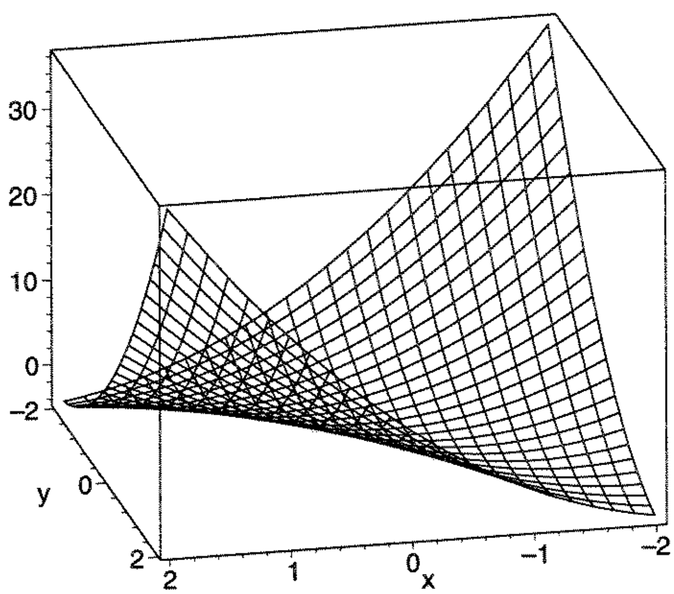


Math 2270-1
Wednesday November 30
Some pictures to go with our computations:

```
> with(plots):
> implicitplot(2*x^2+2*y^2+5*x*y=1,x=-3..3,y=-3..3,
  color=black);
```




```
> plot3d(2*x^2+2*y^2+5*x*y,x=-2..2,y=-2..2,
  axes=boxed,style=wireframe,color=black);
```




```
>
```

Big picture:

In order to understand the quadratic form

$$Q(\vec{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \vec{x}^T A \vec{x}$$

(may take $A$ symmetric)
we take $A$ real

$A$ symm $\Rightarrow \exists$ o.n. eigenbasis $\mathcal{B} = \{\vec{w}_1, \vec{w}_2 \ldots \vec{w}_n\}$

so $S = [\vec{w}_1 | \vec{w}_2 | \cdots \vec{w}_n]$ is orthog matrix

so $D = S^T A S = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$$\vec{x} = S[\vec{x}]_{\mathcal{B}} = S\vec{x}'$$

$$\vec{x}^T A \vec{x} = [\vec{x}']^T \underbrace{S^T A S}_{D} [\vec{x}']$$

$$= \sum_{i=1}^{n} \lambda_i (x_i')^2$$

applications to conics, quadrics, $2^{nd}$ deriv test, $2^{nd}$ order behavior of fcns

Steps ① If $A$ is symmetric all evals are real! (charact poly factors over reals!)

② $A$ symmetric $\Rightarrow A$ diagonalizable

③ $A$ symmetric, $A\vec{v} = \lambda_1 \vec{v}$, $A\vec{w} = \lambda_2 \vec{w}$, $\lambda_1 \neq \lambda_2 \Rightarrow \vec{v} \perp \vec{w}$

④ $A$ symmetric $\Rightarrow \exists S$ orthog s.t. $S^T A S = D$, $D$ diagonal.

this is called the spectral thm

We'll do this next week! (Monday) You do the $A_{2\times 2}$ case by complete brute force, to engender confidence that the general case might work...

the general proof will shed more light on the situation