Math 2270-1
Tuesday 11/29

§7.6 Stability

Example $\vec{x}(t+1) = A\vec{x}(t)$

if $A_{n\times n}$ diagonalizable (at least over $\mathbb{C}$)

then $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a complex eigenbasis, evals $\lambda_1, \lambda_2, \ldots \lambda_n$

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$A^t\vec{x}_0 = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \cdots + c_n\lambda_n^t\vec{v}_n$$

$$\lambda_k = r_k e^{i\theta_k} \qquad r_k = |\lambda_k|,\ \theta_k = \text{polar angle}$$

$$\lambda_k^t = r_k^t e^{it\theta_k}$$

Deduce: if each $|\lambda_k| < 1$ then $A^t\vec{x}_0 \to \vec{0}$ "geometrically" as $t \to \infty$.

if at least one $|\lambda_k| > 1$ then $\exists\ \vec{x}_0$ with $\|\vec{x}_0\|$ arbitrarily small so that $\|A^t\vec{x}_0\| \to \infty$ as $t\to\infty$.

Def $\vec{0}$ is an asymptotically stable equilibrium of a discrete dynamical system

$$\vec{x}(t+1) = A\vec{x}(t)$$

iff $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$ for all $\vec{x}_0$.

[think of the Glucose-Insulin model, in which $\vec{x} = \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$ measured deviations from equilibrium].

if this is all you need to know, the test is easy

Theorem : For the discrete dynamical system

$$\vec{x}(t+1) = A\vec{x}(t)$$

$\vec{0}$ is an asymptotically stable equilibrium iff all evals of A satisfy $|\lambda| < 1$.

pf : Well, we just did the proof in case of diagonalizable A.
The general proof relies on the Jordan canonical form of A, in case A is not diagonalizable. (We will talk about Jordan form in 2250)

Examples:

1 Since the only $A_{2\times 2}$ for which we don't know the stability thm is the case of equal eigenvalues, let's focus there:

$$|A-\lambda I| = \lambda^2 - \mathrm{trace}(A)\lambda + |A|$$
$$= \left(\lambda - \tfrac{1}{2}\mathrm{trace}(A)\right)^2 \qquad (\text{discrim} = 0)$$
$$= (\lambda-\lambda_1)^2$$

Case 1 $\dim E_{\lambda_1} = 2$: this is the diagonalizable case A similar to $\lambda_1 I$ $(\Rightarrow A = \lambda_1 I)$

Case 2 $\dim E_{\lambda_1} = 1$

let $E_{\lambda_1} = \mathrm{span}\{\vec{u}\}$.

Let $\mathcal{B} = \{\vec{u}, \vec{v}\}$ be a basis for $\mathbb{R}^2$

$$L(\vec{x}) = A\vec{x}$$

$$B = [L]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & b_1 \\ 0 & b_2 \end{bmatrix}$$ but $|B-\lambda I|$ also $= (\lambda-\lambda_1)^2$ $\Rightarrow b_2 = \lambda_1$

$$= \begin{bmatrix} \lambda_1 & b_1 \\ 0 & \lambda_1 \end{bmatrix}$$, $b_1 \neq 0$ since $\dim(E_{\lambda_1}) = 1$.

$$\Rightarrow L(\vec{v}) = b_1\vec{u} + \lambda_1 \vec{v}$$
$$\Rightarrow L\left(\tfrac{1}{b_1}\vec{v}\right) = \vec{u} + \lambda_1\left(\tfrac{\vec{v}}{b_1}\right)$$

let $\mathcal{C} = \{\vec{u}, \tfrac{1}{b_1}\vec{v}\}$

$$C = [L]_{\mathcal{C}} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$ this is known as a 2×2 Jordan block. [is composition of scaling & shear].

$$C = S^{-1}AS$$
$$A = SCS^{-1}$$
$$A^t = SC^tS^{-1}$$

$$C = \lambda_1 I + N \qquad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C^n = \lambda_1^n I + n\lambda_1^{n-1}N + 0 + 0 + 0 \cdot 0 \qquad N^2 = 0$$

(binomial expansion)

[matrices for which a power of the matrix is zero are called nilpotent].

$$A^t = S\left[\lambda_1^t I + n\lambda_1^{t-1}N\right]S^{-1}$$

$$\to 0 \text{ if } |\lambda_1| < 1$$

② The general case of complex roots for real $A_{2\times 2}$ [What happens with glucose-insulin was typical]

$$A\vec{w} = \lambda\vec{w} \qquad \lambda = p + iq \qquad (\text{assume } b \neq 0)$$
$$A\bar{\vec{w}} = \bar{\lambda}\bar{\vec{w}} \qquad \bar{\lambda} = p - iq$$

$$\vec{w} = \vec{u} + i\vec{v}$$
$$A\vec{w} = (p+iq)(\vec{u}+i\vec{v})$$
$$A\vec{u} + iA\vec{v} = (p\vec{u} - q\vec{v}) + i(p\vec{v} + q\vec{u}) \quad ; \quad \text{So } \begin{array}{l} A\vec{u} = p\vec{u} - q\vec{v} \\ A\vec{v} = p\vec{v} + q\vec{u} \end{array}$$

for $\mathcal{B} = \{v, u\}$

$L(\vec{x}) = A\vec{x}$

$$B = [L]_{\mathcal{B}} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$$ ← rotation-dilation

$$= r\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad ; \quad r = \sqrt{p^2+q^2}$$

$$B = S^{-1}AS$$
$$A = SBS^{-1}$$
$$A^t = SB^tS^{-1}$$
$$= r^t S \begin{bmatrix} \cos(t\phi) & -\sin(t\phi) \\ \sin(t\phi) & \cos(t\phi) \end{bmatrix} S^{-1} \qquad t \in \mathbb{Z}$$

$$\underbrace{A^t\vec{x}_0 = r^t \underbrace{S \underbrace{\begin{bmatrix} \cos(t\phi) & -\sin(t\phi) \\ \sin(t\phi) & \cos(t\phi) \end{bmatrix} \underbrace{S^{-1}\vec{x}_0}_{[\vec{x}_0]_{\mathcal{B}}}}_{\text{points on a circle of radius } \|S^{-1}\vec{x}_0\|}}_{\text{converts circle to an ellipse}}}_{}$$

scales points to spiral
outward if $r > 1$
inward if $r < 1$
stays ellipse if $r = 1$

$$r = \sqrt{p^2+q^2} = |\lambda|.$$

Let's revisit glucose-insulin from this point of view!

$$A = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix}$$

$$\lambda = p + iq$$
$$= .9 + .2i$$

$$\vec{w} = \begin{bmatrix} -2 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{= \vec{u}} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= \vec{v}.}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$$

check!

$$B = [L]_{\mathcal{B}} = \begin{bmatrix} .9 & -.2 \\ .2 & .9 \end{bmatrix} = \sqrt{.85} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\phi = \arctan(2/9)$$
$$r = \sqrt{.85}$$

$$S = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix}$$

$$A^t \vec{x}_0 = r^t \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(t\phi) & -\sin(t\phi) \\ \sin(t\phi) & \cos(t\phi) \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 0 \\ -50 \end{bmatrix}}$$

$$\begin{bmatrix} 50 \sin(t\phi) \\ -50 \cos(t\phi) \end{bmatrix}$$

on circle $x^2 + y^2 = 50^2$

$$\begin{bmatrix} 100 \cos(t\phi) \\ 50 \sin(t\phi) \end{bmatrix}$$

on ellipse $\frac{x^2}{100^2} + \frac{y^2}{50^2} = 1.$

$$(.85)^{t/2} \begin{bmatrix} 100 \cos(t\phi) \\ 50 \sin(t\phi) \end{bmatrix}$$

spiral

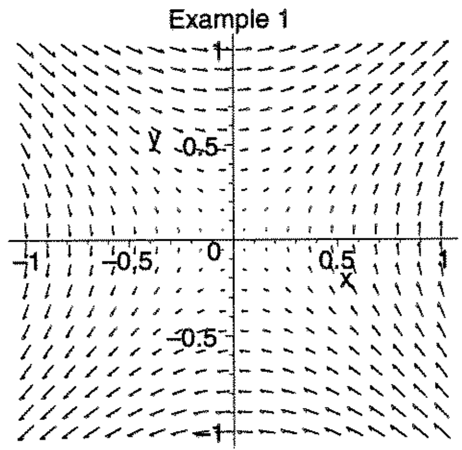
Math 2270-1
November 29, 2005
Phase portraits, eigenvectors and eigenvalues, stability

```
> with(plots):with(linalg):
> A:=matrix(2,2,[1,2,2,1]);
  eigenvects(A);
```

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

[3, 1, {[1, 1]}], [-1, 1, {[-1, 1]}]

```
> pict1:=fieldplot([0*x+2*y,2*x+0*y],x=-1..1,y=-1..1):
  display({pict1},title=`Example 1`);
```

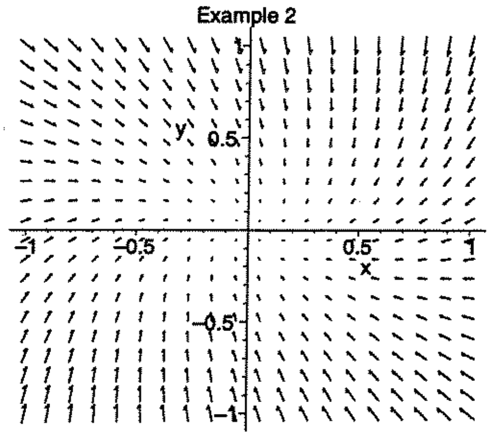



```
> B:=matrix(2,2,[-1,1,-1,-3]);
  eigenvects(B);
```

$$B := \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$$

[-2, 2, {[1, -1]}]

```
> pict2:=fieldplot([-2*x+y,-1*x-4*y],x=-1..1,y=-1..1):
  display({pict2},title=`Example 2`);
```

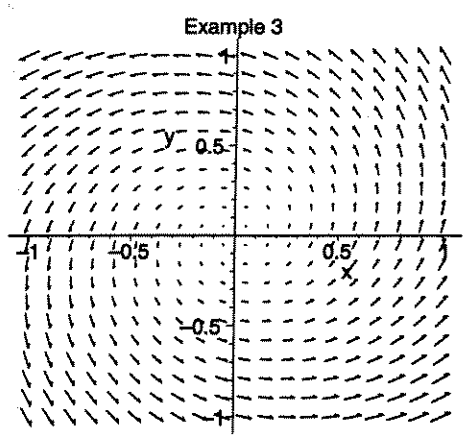



```
> C:=matrix(2,2,[2,-3,3,2]);
  eigenvects(C);
```

$$C := \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$[2+3I, 1, \{[1, -I]\}], [2-3I, 1, \{[1, I]\}]$$

```
> pict3:=fieldplot([x-3*y,3*x+y],x=-1..1,y=-1..1):
  display({pict3},title='Example 3');
```




```
>
```