HW for Fri 12/2

7.5 1 2 3 4 5 6 9 11 21 24 30 31 32 41 45 47

7.6 1 3 4 11 12 17 20 37

Chapter 7 Review True-False, multiples of 6

Math 2270-1

Wednesday Nov 23

MAPLE LAB WILL BE FRIDAY NEXT WEEK, NOT MONDAY

- Glucose-insulin example Tuesday notes!

We need to review and better-understand complex number arithmetic

$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$

If $\vec{z} := a + bi$, $\vec{w} := c + di$

$\vec{z} = \vec{w}$ iff $a = c$ & $b = d$

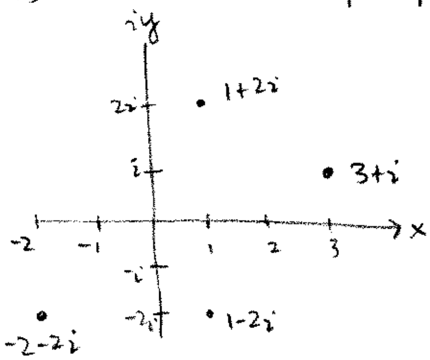
then $\vec{z} + \vec{w} := (a+c) + (b+d)i$

Notice this makes $\mathbb{C}$ a real vector space (i.e. real scalars), of dim 2

With respect to the natural basis $\beta = \{1, i\}$ the coordinate map gives (the usual) isomorphism to $\mathbb{R}^2$

$$\vec{z} = a + bi, \qquad [\vec{z}]_\beta = \begin{bmatrix} a \\ b \end{bmatrix}.$$

This leads to the "complex plane" representation for complex numbers:



Interesting geometry starts happening when you combine the geometry of the complex plane with algebraic operations such as complex multiplication and conjugation

<u>Def</u> For $z = a + bi$, $w = c + di$

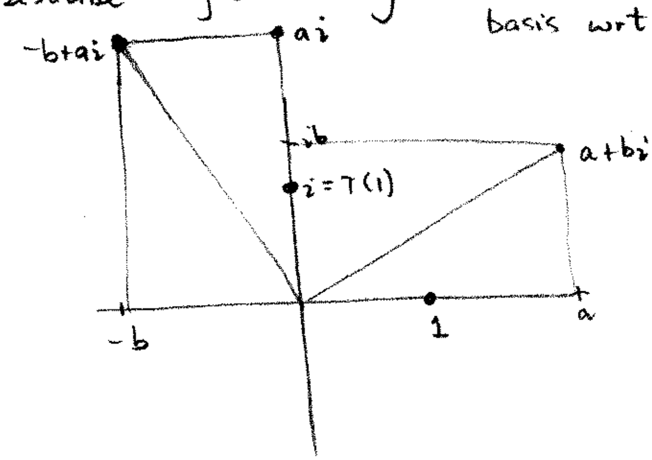
$$zw = (a+bi)(c+di) := (ac - bd) + i(bc + ad)$$

[because we define $i^2 = -1$]

<u>check</u>: $zw = wz$.

<u>example</u> Let $T(z) = iz$. Describe T geometrically. (T is linear. What is its basis wrt $\{1, i\}$?)

$i(a+bi) = -b + ai$



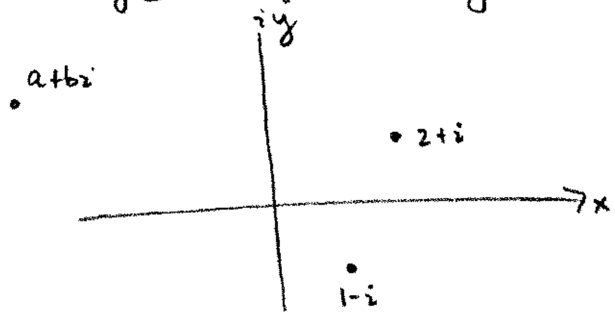
<u>Description</u>:

Another operation on complex numbers is <u>conjugation</u>

<u>Def</u> let $z = a+bi \in \mathbb{C}$ (a, b real)

$$\bar{z} := a - bi$$

Describe conjugation geometrically:



<u>Def</u> let $z = a+bi$

$$|z|^2 := a^2 + b^2 = z\bar{z}$$

also, <u>check</u>: $\overline{zw} = \bar{z}\,\bar{w}$, so also $|zw| = |z||w|$

<u>check</u>: $zw = 0$ iff z or $w = 0$

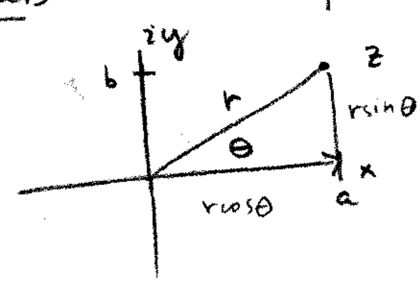
<u>check</u> mult inv $\frac{1}{z}$ exists $\forall z \neq 0$, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

<u>Polar form for complex numbers</u> ⟷ corresponds to polar coords in the plane

let $z = a+bi$

then

$$z = r\cos\theta + i r\sin\theta$$



where $r = |z|$ and θ are the usual polar coords of $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

<u>Multiplication</u>!

if also $w = \rho(\cos\phi + i\sin\phi)$

then $zw = r\rho\left[(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\right]$

$$= \underbrace{r\rho}_{\text{new modulus.}}\Big[\underbrace{\cos\theta\cos\phi - \sin\theta\sin\phi}_{\cos(\theta+\phi)} + i\big(\underbrace{\cos\theta\sin\phi + \sin\theta\cos\phi}_{\sin(\theta+\phi)}\big)\Big]$$

moduli multiply
angles add] !

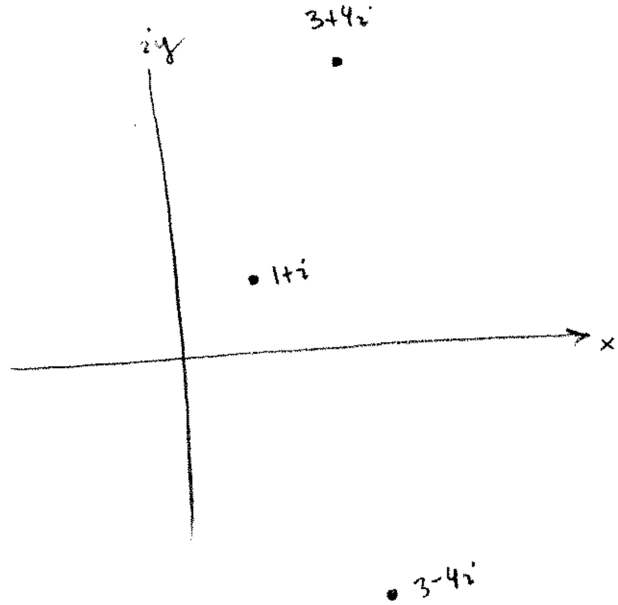
play with multiplication

$(3-4i)(3+4i)$

exple of $z\bar{z}$

$(1+i)^2$, $(1+i)^3$

others?



Tie-in to Euler's formula:

Remember (from Calculus)(?) Euler's formula:

$$e^{i\theta} := \cos\theta + i\sin\theta$$

Taylor Series $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots$

plug in $e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots$

$\cos\theta$! $i\sin\theta$!

using this notation, if

$z = re^{i\theta}$

$w = \rho e^{i\phi}$

then $zw = r\rho e^{i(\theta+\phi)}$; makes sense!

(follows from computation on page 2)

Example : Let $T(z) = (1+i)z$ geometrically

① Since $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}e^{i\pi/4}$

$T(z)$ must dilate by a factor of $\sqrt{2}$ and rotate by $\pi/4$

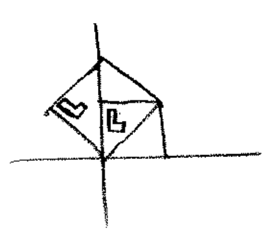
② $T(a+bi) = (1+i)(a+bi)$

$= (a-b) + i(a+b)$

$$[T]_{\mathcal{B}} = A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

a rotation-dilation matrix!

✓



Return to glucose-insulin model

$$\begin{bmatrix} G(t+1) \\ H(t+1) \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} G(t) \\ H(t) \end{bmatrix} ; \qquad \begin{cases} \vec{x}(t) = A^t \vec{x}(0) \\ \vec{x}(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \end{cases}$$

we did.

$$|A - \lambda I| = \begin{vmatrix} (.9-\lambda) & -.4 \\ .1 & (.9-\lambda) \end{vmatrix} = (\lambda - .9)^2 + .04$$
$$= (\lambda - .9 - .2i)(\lambda - .9 + .2i)$$

roots $\lambda = .9 \pm .2i$

$\lambda = .9 + .2i$

$$\begin{array}{l|cc|c} & -.2i & -.4 & 0 \\ & 1 & -.2i & 0 \\ \hline -10R_2 & 1 & -2i & 0 \\ -R_1 & .2i & .4 & 0 \\ \hline & 1 & -2i & 0 \\ -.2iR_1 + R_2 & 0 & 0 & 0 \end{array} \quad !$$

$\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ $\leftarrow$ for IVP prefer $\boxed{\lambda = .9 + .2i \quad \begin{bmatrix} -2 \\ i \end{bmatrix}}$

$$A\vec{v} = \lambda\vec{v} \Rightarrow \overline{A\vec{v}} = A\bar{\vec{v}}$$
$$\overline{\lambda\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

(the conjugate eigenvalue has conjugate eigenvector)

so for $\lambda = .9 - .2i$
$\vec{v} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$

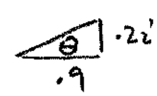
for IVP prefer $\boxed{\lambda = .9 - .2i \quad \vec{v} = \begin{bmatrix} 2 \\ i \end{bmatrix}}$

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = -25 \begin{bmatrix} -2 \\ i \end{bmatrix} + 25 \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$$A^t \begin{bmatrix} 100 \\ 0 \end{bmatrix} = -25\,(.9 + .2i)^t \begin{bmatrix} -2 \\ i \end{bmatrix} + 25\,(.9 - .2i)^t \begin{bmatrix} 2 \\ i \end{bmatrix}$$

$(.9+.2i)^t = (re^{i\theta})^t = r^t e^{i\theta t}$ $\qquad (.9-.2i)^t = (re^{-i\theta})^t$

$$\boxed{r = \sqrt{.85} \qquad \theta = \arctan(2/9)}$$



$$= 25\left[(.85)^{t/2}\right]\left[[\cos(t\theta) + i\sin(t\theta)]\begin{bmatrix} 2 \\ -i \end{bmatrix} + [\cos(t\theta) - i\sin(t\theta)]\begin{bmatrix} 2 \\ i \end{bmatrix}\right]$$

$$\boxed{= 25(.85)^{t/2} \begin{bmatrix} 4\cos(t\theta) \\ 2\sin(t\theta) \end{bmatrix}}$$

needs a picture, but the formula does indicate what happens.

Can we get a nice, closed form for the solution? Let's try our eigenvalue, eigenvector approach:

```
> eigenvects(A);
```

$$[.9+.2000000000\,I,\ 1,\ \{[-2.000000000,\ I]\}],\ [.9-.2000000000\,I,\ 1,\ \{[-2.000000000,\ -I]\}]$$

```
>
```

Can we use this to find a closed form solution to the problem?.....
In class notes we have done so! Here is the solution we have found:

```
> r:=sqrt(.85);
  theta:=arctan(.2/.9);
  G:=t->100*r^t*cos(theta*t);
  H:=t->50*r^t*sin(theta*t);
```

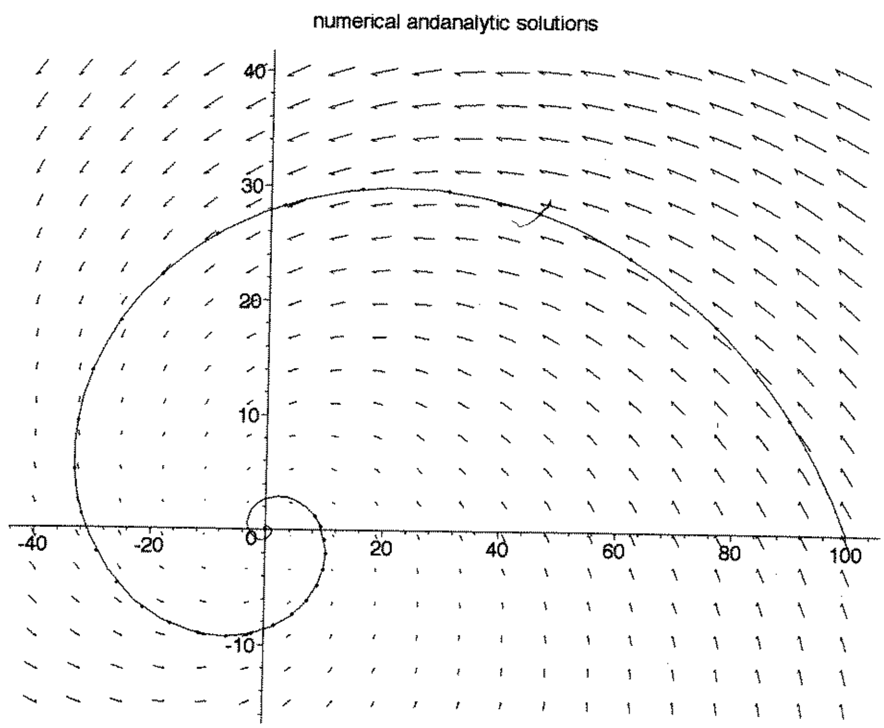
$$r := .9219544457$$

$$\theta := .2186689459$$

$$G := t \rightarrow 100\, r^t \cos(\theta\, t)$$

$$H := t \rightarrow 50\, r^t \sin(\theta\, t)$$

```
> actual:=plot([G(t),H(t),t=0..100],color=black):
  display({actual,pict1,soltn},title=`numerical and
  analytic solutions`);
```

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