Math 2270-1
Monday 21 Nov.

§7.3-7.4 : Similar matrices, eigenvalues, eigenvectors & diagonalizability.

- Discussion on page 1 Wednesday (?!) about why similar matrices have
  - identical characteristic polynomials
    hence identical eigenvalues, with <u>equal algebraic multiplicities</u>
  - corresponding eigenvalues λ_i also have <u>equal geometric multiplicities</u>

We did some of the page 2 Wed examples on Friday, but left a question hanging...
(the <u>Note</u> below might help us...)

<u>Note</u>: in general, even if two matrices have the same characteristic polynomial, and if their eigenvalues also yield equal geometric multiplicities, the matrices may not be similar, eg. §7.4 HW #54 is such an example.

<u>BUT</u>: If A and B (are $n\times n$), and have <u>distinct</u> eigenvalues $\lambda_1, \lambda_2, .. \lambda_n$
then A has a corresponding eigenbasis $\mathcal{A} = \{\vec{v}_1, \vec{v}_2, - \vec{v}_n\}$
s.t. for $T_1\vec{x} = A\vec{x}$,

$$[T]_{\mathcal{A}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} = S_1^{-1} A S_1 \qquad \left(S_1 = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right]\right) = S_{E \leftarrow \mathcal{A}}$$

but B has an eigenbasis $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, .. \vec{w}_n\}$
s.t. for $T_2\vec{x} = B\vec{x}$

$$[T_2]_{\mathcal{B}} = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\substack{\shortparallel \\ \Lambda}} = S_2^{-1} B S_2$$

so
$$\begin{aligned} A &= S_1 \Lambda S_1^{-1} \\ &= S_1 S_2^{-1} B S_2 S_1^{-1} \\ &= (S_2 S_1^{-1})^{-1} B (S_2 S_1^{-1}) \end{aligned}$$

– so –
A and B are similar!

- Discussion of "diagonalizable" matrices, page 3 Friday

(example: any $n\times n$ matrix with n distinct eigenvalues)

The precise theorem about when $A_{n\times n}$ is diagonalizable...

<u>Theorem</u>: Let $|A-\lambda I| = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$ $\quad k_1+k_2+\cdots+k_\ell=n$, λ_i's distinct

Then

- $\dim(E_{\lambda_i}) \le k_i \quad i=1,2,\ldots \ell$

  (geom mult of $\lambda_i \le$ alg mult)

- A is diagonalizable if and only if $\dim(E_{\lambda_i}) = k_i \quad i=1,2..\ell$

  In this case an eigenbasis for A may be constructed by amalgamating bases for each E_{λ_i}; such a collection of n eigenvectors will always be linearly independent, so a basis for $\mathbb{R}^n$.

<u>proof</u>: • $\dim(E_{\lambda_i}) \le k_i$:

Let $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\} = \mathcal{B}$ a basis for E_{λ_i}, extend to a basis $\{\vec{v}_1, \ldots \vec{v}_k, \vec{v}_{k+1}, \ldots \vec{v}_n\}$ for $\mathbb{R}^n$

Let $T(\vec{x}) = A\vec{x}$, let $B = [T]_{\mathcal{B}}$.

$$\text{Then } B = \left[\begin{array}{c|c} \begin{matrix}\lambda_i & 0 & O\\ 0 & \lambda_i & \\ O & & \lambda_i\end{matrix} & \text{mess} \\ \hline O & \end{array}\right]$$

$\underbrace{\qquad}_{k}$

So $|B-\lambda I| = (\lambda_i - \lambda)^k$ (stuff) $= |A-\lambda I|$ because B & A similar

(expand down left columns, successively)

$\Rightarrow k \le k_i$

(because $|A-\lambda I|$ has only $(\lambda-\lambda_i)^{k_i}$ factor) ∎

$\Rightarrow$ • If A <u>is</u> diagonalizable, then A is similar to a diagonal matrix having A's characteristic poly – i.e. λ_i occurs in exactly k_i columns of $\Lambda = S^{-1}AS$. Thus there are exactly k_i columns of S which are eigenvectors with eigenvalue λ

$\Rightarrow \dim(E_{\lambda_i}) \ge k_i$; But reverse inequality from first bullet, so $\dim(E_{\lambda_i}) = k_i$

$\Leftarrow$ • If $\dim(E_{\lambda_i}) = k_i$, $i=1,2,\ldots\ell$

pick $\{\vec{v}_{i1}, \vec{v}_{i2}, \ldots \vec{v}_{ik_i}\}$ = basis E_{λ_i}, and amalgamate these bases to get n vectors in $\mathbb{R}^n$.

We claim these n vectors are independent:

$$\text{If } \underbrace{(c_{11}\vec{v}_{11} + \cdots + c_{1k_1}\vec{v}_{1k_1})}_{\in E_{\lambda_1},\ =\vec{w}_1} + \underbrace{(c_{21}\vec{v}_{21} + \cdots + c_{2k_2}\vec{v}_{2k_2})}_{\in E_{\lambda_2},\ =\vec{w}_2} + \cdots + \underbrace{(\qquad)}_{\in E_{\lambda_\ell},\ =\vec{w}_\ell} = \vec{0}$$

$\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_\ell = \vec{0}$ $\quad$ ℓ eigenvects w/ distinct evals

By our earlier thm about eigenvectors with distinct eigenvalues, each $\vec{w}_i$ must equal zero

So each $c_{i1}\vec{v}_{i1} + \cdots + c_{ik_i}\vec{v}_{ik_i} = 0 \Rightarrow c_{i1} = \cdots = c_{ik_i} = 0 \quad \forall i$

$\blacksquare$

<u>Def</u>: Let $T: V \to V$ be a linear transformation.

$\vec{v} \in V$ ($\vec{v} \neq 0$) is an eigenvector of $\vec{v}$ with eigenvalue λ

iff $T(\vec{v}) = \lambda \vec{v}$.

Notice this is true iff $[T(\vec{v})]_{\mathcal{B}} = \lambda [\vec{v}]_{\mathcal{B}}$ for any basis $\mathcal{B}$,

i.e. iff $[\vec{v}]_{\mathcal{B}}$ is an eigenvector of $B = [T]_{\mathcal{B}}$.

Similarly, T is diagonalizable iff $\exists$ an eigenbasis wrt T, for V.

<u>example</u>: $V = \text{span}\{\cos t, \sin t\}$

$T(f) = f'(t)$

Is T diagonalizable?