Math 2270-1
Fri Nov 18

Hw for Wed Nov 23
§7.4 (14, 17), 23, (25, 27, 31, 35, 37, 38, 47, 54)

- Discussion of similar matrices, eigenvalues, eigenvectors, pages 1-2 Wed notes

- Extended discussion of §7.3 #38:

<u>Theorem</u> Let $T(\vec{x}) = A\vec{x}$ where A is an orthogonal matrix $\left(\Leftrightarrow \|T\vec{x}\| = \|\vec{x}\| \ \forall \vec{x} \right.$ $\Leftrightarrow A^T = A^{-1}$ $\Leftrightarrow$ cols of A are ortho-normal$\left.\right)$

Recall $A^T A = I \Rightarrow \det A = \pm 1.$

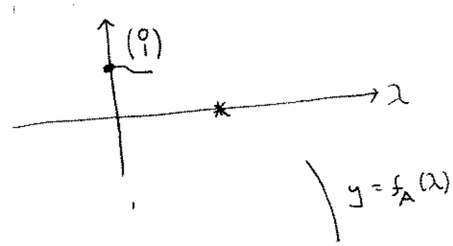
In case $\det A = +1$ there is an axis $\vec{v} \neq 0$ so that
$A\vec{v} = \vec{v}$
and A is rotation about this axis

(In case $\det A = -1$ there is an "axis" $\vec{v} \neq 0$ so that
$A\vec{v} = -\vec{v}$
and A is the composition of rotation about this axis, with reflection through the subspace (plane) $\perp$ to $\vec{v}$.)

proof: $p_\lambda(A) = |A - \lambda I| = -\lambda^3 + \text{trace}(A)\lambda^2 - (|A_{1,1}| + |A_{2,2}| + |A_{3,3}|)\lambda + |A|$

book's notation! $\to f_A(\lambda) \longrightarrow f_A(0) = |A| = 1$

$$\lim_{\lambda \to \infty} f_A(\lambda) = \lim_{\lambda \to -\infty} -\lambda^3\left[1 + \frac{c_1}{\lambda} - \frac{c_2}{\lambda^2} + \frac{c_3}{\lambda^3}\right] = -\infty$$



By the intermediate value theorem for continuous functions, $\exists\ \lambda_1$ s.t. $f_A(\lambda_1) = |A - \lambda_1 I| = 0.$
$0 < \lambda_1 < \infty.$ Let $\vec{v}$ be an eigenvector for λ_1

But A is orthogonal so
$\|A\vec{v}\| = \|\vec{v}\|$
$\|$
$\|\lambda_1 \vec{v}\|$
$\|$
$\lambda_1 \|\vec{v}\|$

So $\lambda_1 = 1$! (This would "finish" #38).

And $A\vec{v} = \lambda\vec{v}.$

Now, construct an orthonormal basis for $\mathbb{R}^3$, $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \mathcal{B}$

with $\vec{w}_1 = \frac{\vec{v}}{\|\vec{v}\|}$ and $\vec{w}_2, \vec{w}_3$ a basis for $(\text{span}\{\vec{v}\})^\perp$, i.e. the plane thru origin $\perp$ $\vec{v}$

$$A\vec{w}_1 = \vec{w}_1$$

Let B be the matrix for $T(\vec{x}) = A\vec{x}$ w.r.t. $\mathcal{B}$

- $B = S^{-1}AS$

  But $S = S_{E \leftarrow \mathcal{B}} = \left[\vec{w}_1 \middle| \vec{w}_2 \middle| \vec{w}_3 \right]$ is orthogonal, so $S^{-1} = S^T$

$\Rightarrow$ $B = S^TAS$ is also an orthogonal matrix, since

and $|B| = |A| = +1$

$$B^TB = (S^TAS)^T(S^TAS) = I \; !$$
$$= S^T A^T \underbrace{(S^T)^T S^T}_{I} AS$$
$$= I$$

also,

$$B = \left[[T(\vec{w}_1)]_{\mathcal{B}} \middle| [T(\vec{w}_2)]_{\mathcal{B}} \middle| [T(\vec{w}_3)]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 1 & ? & ? \\ 0 & & \\ 0 & & \end{bmatrix}$$

B orthogonal $\Rightarrow$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & ? & ? \\ 0 & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}$, $a^2+b^2=1$, $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$

det $B = +1$, so this rules out other possibility $\begin{bmatrix} 0 \\ b \\ -a \end{bmatrix}$.

a rotation by α around $\vec{v}$! ∎

<u>Example</u>: Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Find the axis & rotation angle!

<u>Def.</u> $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is an <u>eigenbasis</u> for A if each $\vec{v}_i$ is an eigenvector of A, and if $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

<u>Def</u> A is called <u>diagonalizable</u> if it is similar to a diagonal matrix, i.e.

$\exists$ invertible matrix S s.t.

$$B = S^{-1}AS$$

is a diagonal matrix $B = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$

Notice, A is diagonalizable if and only if it has an eigenbasis:

- If $\{\vec{v}_1, \dots \vec{v}_n\} = \mathcal{B}$ is an eigenbasis, then for $T(\vec{x}) = A\vec{x}$

$$B = [T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = S^{-1}A\underset{\underset{\left[\vec{v}_1 | \vec{v}_2 | \cdots \vec{v}_n\right]}{\shortparallel}}{S}$$

- If $\underset{\underset{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}{\shortparallel}}{B} = S^{-1}AS$ then $SB = AS$

so $\operatorname{col}_j(SB) = \operatorname{col}_j(AS) = A\operatorname{col}_j(S)$

$\operatorname{col}_j(SB) = S\operatorname{col}_j(B) = \lambda_j \operatorname{col}_j(S)$

so the columns of S are an eigenbasis.

<u>Theorem</u> (for Monday!)

Let $|A - \lambda I| = \pm(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$ $\quad k_1 + k_2 + \dots + k_\ell = n$, λ_i's distinct.

- Then $\dim(E_{\lambda_i}) \le k_i \quad i = 1, 2, \dots \ell$)

  (geom. mult $\le$ alg mult.)

- A is diagonalizable iff $\dim(E_{\lambda_i}) = k_i \quad , i = 1, 2, \dots \ell$

  And in this case, an eigenbasis for A may be constructed by amalgamating bases for each E_{λ_i}.

  (such a collection of n eigenvectors will always be linearly ind., hence a basis for $\mathbb{R}^n$)