Math 2270-1
Wed Nov. 16

Finish Tuesday notes, also look at pages 3-4 today about the characteristic polynomial for nxn matrices
- page 4 example
- page 5 theorem
- page 6 example.

Notation:

If $\lambda_i$ is an eigenvalue of $A$, $E_{\lambda_i}(A)$ denotes the $\lambda_i$-eigenspace.
$\dim(E_{\lambda_i}(A))$ is called the <u>geometric multiplicity</u> of $\lambda_i$.

If $p_\lambda(A)$ is the characteristic polynomial of $A$, $\det(A-\lambda I)$,
and $p_\lambda(A) = (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$ (always factors over the complex #'s)

$\lambda_1, \lambda_2, \ldots \lambda_\ell$ distinct

then $k_i$ is the <u>algebraic multiplicity</u> of $\lambda_i$

<u>Similar matrices, eigenvalues, eigenvectors</u>

Let $A$ and $B$ be similar,
$B = S^{-1}AS$ $\quad (A = SBS^{-1})$

Then
- $p_\lambda(A) = p_\lambda(B)$: (so the eigenvalues $\lambda_i$ & <u>algebraic multiplicities</u> of evals for $A$ & $B$ are the same!)

$$|B-\lambda I| = |S^{-1}AS - \lambda I| = |S^{-1}AS - S^{-1}(\lambda I)S| = |S^{-1}(AS-\lambda IS|$$
$$= |S^{-1}(A-\lambda I)S|$$
$$= |S^{-1}||A-\lambda I||S|$$
$$= |A-\lambda I|$$

- The $\mathbb{R}^n$ isomorphism $T\vec{v} = S^{-1}\vec{v}$
  restricts to an isomorphism
  $$T: E_{\lambda_i}(A) \rightarrow E_{\lambda_i}(B)$$
  (So the <u>geometric multiplicities</u> of $\lambda_i$ are the same for $A$ & $B$)

<u>proof</u>: If $\vec{v} \in E_{\lambda_i}(A)$ then $A\vec{v} = \lambda_i\vec{v}$

and $B(S^{-1}\vec{v}) = S^{-1}A(SS^{-1})\vec{v} = S^{-1}A\vec{v} = S^{-1}\lambda_i\vec{v} = \lambda_i(S^{-1}\vec{v})$

so $S^{-1}\vec{v} \in E_{\lambda_i}(B)$

Similarly, if $\vec{w} \in E_{\lambda_i}(B)$
then $S\vec{w} \in E_{\lambda_i}(A)$ ∎

If $A = [L]_{\mathcal{a}}$
$B = [L]_{\mathcal{B}}$
then $S = S_{\mathcal{a}\leftarrow\mathcal{B}}$
$S^{-1} = S_{\mathcal{B}\leftarrow\mathcal{a}}$

Examples:

We showed on Tuesday that

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad \text{is similar to} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

- is $A$ similar to $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ?

- is $A$ similar to $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ ?

- is $A$ similar to $\begin{bmatrix} 2 & 1 \\ 7 & 1 \end{bmatrix}$ ?

Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 7 & 8 & 1 \end{bmatrix}$. Are $C$ and $D$ similar?

from page 5 Tuesday:

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Are $A$ and $C$ similar?

[redacted]

# Determinants and the characteristic polynomial

- Determinants, another expression for them!
  (See pages 268–269)

<u>Theorem</u> Let $A_{n\times n}$. Then

$$\det A = \sum_{\substack{\sigma = j_1 j_2 \cdots j_n \\ \text{a permutation} \\ \text{of } 1\,2 \cdots n}} \underbrace{\text{sign}(\sigma)}_{\pm 1}\, a_{1 j_1} a_{2 j_2} \cdots a_{n j_n}$$

$$\text{sign}(\sigma) = \begin{cases} +1 & \text{if there are an even number of inversions in the permutation } j_1 j_2 \cdots j_n \\ -1 & \text{if the number of inversions is odd.} \end{cases}$$

instances where $j_i > j_\ell$ but $i < \ell$ (later column comes before earlier column)

$$|A|_{2\times 2} = a_{11}a_{22} - a_{12}a_{21}$$

$\begin{bmatrix} o & \\ & o \end{bmatrix}$ $\sigma = 1\ 2$, # inversions = 0

$\begin{bmatrix} & o \\ o & \end{bmatrix}$ $\sigma = 2\ 1$, # of inversions = 1

$|A|_{3\times 3}$:

$\begin{bmatrix} o & & \\ & o & \\ & & o \end{bmatrix}$ $1\ 2\ 3$, $+1$

$\begin{bmatrix} & o & \\ & & o \\ o & & \end{bmatrix}$ $2\ 3\ 1$, $+1$

$\begin{bmatrix} & & o \\ o & & \\ & o & \end{bmatrix}$ $3\ 1\ 2$, $+1$

$\begin{bmatrix} & & o \\ & o & \\ o & & \end{bmatrix}$ $3\ 2\ 1$, $-1$

$\begin{bmatrix} o & & \\ & & o \\ & o & \end{bmatrix}$ $1\ 3\ 2$, $-1$

$\begin{bmatrix} & o & \\ o & & \\ & & o \end{bmatrix}$ $2\ 1\ 3$, $-1$

Proof by induction!
picture: Expand across $(n\times n)$ matrix across top row, assume thm works for $(n-1)\times(n-1)$ matrices.

$\begin{bmatrix} \square & \square & \square & \cdots & \square \\ & & & & \\ & & & & \end{bmatrix}$

$$|A| = a_{11}|A_{1,1}| - a_{12}|A_{1,2}| + \overbrace{a_{13}|A_{1,3}|}^{\text{2 extra inversions!}} + \cdots + \overbrace{(-1)^n a_{1n}|A_{1,n}|}^{n \text{ extras!}}$$

$$= a_{11} \sum_{\tilde\sigma = j_2 j_3 \cdots j_n} (\text{sign}\,\tilde\sigma)\, a_{2 j_2} a_{3 j_3} \cdots a_{n j_n}$$

yields $\sigma = 1\, j_2 \cdots j_n$
no extra inversions

$$- a_{12} \sum_{j_2 j_3 \cdots n} (\text{sign}\,\tilde\sigma)\, a_{2 j_2} a_{3 j_3} \cdots a_{n j_n}$$

yields $\sigma = 2\, j_2 j_3 \cdots j_n$
1 extra inversion

Corollary (General expression for $\det(A-\lambda I)$!)

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & & \\ \vdots & & & \ddots & \\ a_{n1} & a_{n2} & & & a_{nn}-\lambda \end{bmatrix}$$

- $p_n(\lambda)$ is a polynomial of degree $n$.
- More precisely,

$$p_n(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{trace}(A) \lambda^{n-1} + (-1)^{n-2} \left( \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \right) \lambda^{n-2}$$

from $\sigma = 1, 2, \cdots n$, choosing $(-\lambda)$ from each term

from $\sigma = 1, 2, \cdots n$ choose $(n-1)$ $-\lambda$'s, and one $a_{ii}$

choose $-\lambda$ from $(n-2)$ diagonal elts, (leaving $i^{th}$ & $j^{th}$ rows & cols)

$$+ \cdots + (-1)^{n-k} \left( \sum \text{all } k\times k \text{ diagonally centered subdeterminants} \right) \lambda^{n-k}$$

$$+ \cdots + (-1)^1 \left( \sum_i |A_{i,i}| \right) \lambda + \det(A)$$

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