Math 2270-1
Tuesday Nov 15

§7.2-7.3: Eigenvalues and eigenvectors

Recall from the coyote-road runner discrete dynamical system example the importance of:

If $A_{n\times n}$ and $\vec{v} \neq \vec{0}$ with $A\vec{v} = \lambda\vec{v}$
then $\vec{v}$ is an eigenvector of A with eigenvalue λ

How to find eigenvalues, then eigenvectors:

$$A\vec{v} = \lambda\vec{v}$$
iff $A\vec{v} - \lambda\vec{v} = \vec{0}$
iff $A\vec{v} - \lambda I\,\vec{v} = \vec{0}$
iff $(A-\lambda I)\vec{v} = \vec{0}$. For $\vec{v} \neq \vec{0}$ this can happen iff $A-\lambda I$ is not invertible
i.e. $\det(A-\lambda I) = 0$

step 1: Compute $\det(A-\lambda I) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}-\lambda \end{bmatrix}$

$\det(A-\lambda I) = p(\lambda)$

called the characteristic polynomial

this is a polynomial in λ of degree n, (prove by induction!)
$$p(\lambda) = (-1)^n \lambda^n + \ldots + \det A$$

Its roots are eigenvalues.

step 2: For each eigenvalue λ the
$\{\vec{v} \mid (A-\lambda I)\vec{v} = 0\}$ is the kernel of $A-\lambda I$
so is a subspace, and has a basis of eigenvectors, which we find in the usual way by computing $\operatorname{rref}(A-\lambda I \mid \vec{0})$ and backsolving.

step 3: In applications we often hope there is a basis of $\mathbb{R}^n$ made out of eigenvectors of A (like in the Coyote-roadrunner problem). This will not always be the case, although under certain conditions on A such a result is guaranteed.

Example : $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

- Find the eigenvalues λ of A

- For each eigenvalue find an eigenbasis for the corresponding eigenspace.

- Is there a basis of $\mathbb{R}^2$ made out of eigenvectors of A?
  What is the matrix of $f(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{x}$ with respect to this basis?
  Find this matrix B two ways!

2x2 case: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &= \lambda^2 - \underset{\substack{\uparrow \\ \text{sum of diagonal elts (also for } n\times n)}}{\text{trace}(A)}\lambda + \det A \end{aligned}$$

e.g. page 2: $\lambda^2 - 4\lambda - 5$

3x3 case: $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$

Can you show

$$\det(A-\lambda I) = -\lambda^3 + (\text{trace}(A))\lambda^2 - (\underset{\substack{\downarrow \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}}{\det(A_{1,1})} + \underset{\substack{\downarrow \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}}}{\det(A_{2,2})} + \underset{\substack{\downarrow \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}}{\det(A_{3,3})})\lambda + \det A \quad ?$$

Any guesses on how this generalizes to the $n\times n$ case?

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

- What are the eigenvalues of A?

- Find eigenbases for each eigenvalue

- Is A similar to a diagonal matrix? (See thm on next page to save work in checking whether you get an $\mathbb{R}^3$ basis of A-eigenvectors.)

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ be eigenvectors of $A_{n\times n}$, with distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_k$.

no two the same.

Then $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ are linearly independent.

proof: Let

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

apply A: $\Rightarrow$ $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \dots + c_k\lambda_k\vec{v}_k = \vec{0}$

A^2 $\quad c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + \dots + c_k\lambda_k^2\vec{v}_k = \vec{0}$

A^{k-1} $\quad c_1\lambda_1^{k-1}\vec{v}_1 + \dots + c_k\lambda_k^{k-1}\vec{v}_k = \vec{0}$

in matrix form:

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 & c_1\lambda_1 & c_1\lambda_1^2 & \cdots & c_1\lambda_1^{k-1} \\ c_2 & c_2\lambda_2 & c_2\lambda_2^2 & & \vdots \\ \vdots & \vdots & \vdots & & \\ c_k & c_k\lambda_k & c_k\lambda_k^2 & & c_k\lambda_k^{k-1} \end{bmatrix} = [\,0\,]$$

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 & & O \\ & c_2 \ddots & \\ O & & c_k \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & & & & \\ 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{k-1} \end{bmatrix} = [\,0\,]$$

transpose of Vandermonde matrix,
its $\det = \prod_{i>j} (\lambda_i - \lambda_j) \neq 0$

So is invertible.
Multiply by its inverse (on the right)

$$\Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 & & O \\ & c_2 \ddots & \\ O & & c_k \end{bmatrix} = [\,0\,]$$

$$\Rightarrow \begin{bmatrix} c_1\vec{v}_1 & c_2\vec{v}_2 & \cdots & c_k\vec{v}_k \end{bmatrix} = [\,0\,] \Rightarrow c_1 = c_2 = \dots = c_k = 0 \quad ! \quad \blacksquare$$

Corollary: If $A_{n\times n}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$, then there is an $\mathbb{R}^n$ basis of eigenvectors of A.

Proof: Each eigenvalue λ_i has at least one non-zero eigenvector $\vec{v}_i$. (Since the λ_i-eigenspace is at least 1-dim'l)
By theorem above, $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ are linearly independent.
since $\dim(\mathbb{R}^n) = n$ they also span $\mathbb{R}^n$ and are a basis.

If A doesn't have n distinct eigenvalues, harder to tell without working:

<u>example</u> $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$p(\lambda) = (2-\lambda)^3$ $p(\lambda) = (2-\lambda)^3$