Math 2270-1
Fri Nov 11

HW: 7.1 1, 2, 4, 6, 7, 15-19, 24, 27, 30, 31, 32, 49
7.2 9, 12, 19, 21, 25, 26, 27, 28, 38
7.3 7, 13, 20, 23, 27, 28, 35, 36, 38, 44

- Summarize geometric meaning of dets (this page)
- Begin §7.1 (page 2)

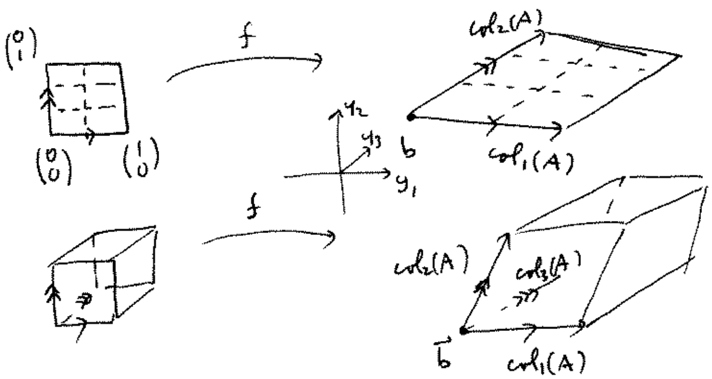
(1) Volume of parallelepiped generated by $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ in $\mathbb{R}^n$ $(k \le n)$

$$Vol = \sqrt{\det(A^T A)} \quad \text{where } A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \middle| \vec{v}_k\right]$$

if $k=n$ (so $A_{n\times n}$) this reduces to

$$Vol = abs(\det(A)) = |\det(A)|$$

(2) If $f(\vec{x}) = A\vec{x} + \vec{b}$, $f: \mathbb{R}^n \to \mathbb{R}^m$ $(m \ge n)$



for any domain set "B"

$$\frac{area(f(B))}{area(B)} = \sqrt{\det(A^T A)} \quad \left(= |\det(A)| \text{ if } m=n \right)$$

$$\frac{vol(f(B))}{vol(B)} = \sqrt{\det(A^T A)} = |\det(A)| \text{ if } m=n.$$

area/volume expansion factors.

(3) If $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ $(m \ge n)$ is differentiable then near any $\vec{x}_0$ $\vec{F}$ is almost affine,

$$F(\vec{x}_0 + \vec{h}) \approx F(\vec{x}_0) + A\vec{h}, \quad A = [F'(x_0)] = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right]$$

the derivative matrix

so the (infinitesimal) volume scaling factor is $\sqrt{A^T A} = |\det(F'(x_0))|$
if $m=n$.

examples:

- $F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ (polar coords) $\left[F'\begin{pmatrix} r \\ \theta \end{pmatrix}\right] = \left[\vec{F}_r \middle| \vec{F}_\theta\right] = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

  $\det = r$. $\quad dA = r\,dr\,d\theta$

- Spherical $F\begin{pmatrix} \rho \\ \theta \\ \phi \end{pmatrix} = \begin{bmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

  $|\det F'| = \rho^2 \sin\phi$

  so $dV = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

- Cylindrical: $F\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$, $\det F' = r$

  so $dV = r\,dr\,d\theta\,dz$

Coyotes & roadrunners §7.1; we will refer to this section

$c(t)$, $r(t)$ pops at year t.

Model:

$$c(t+1) = .86\,c(t) + .08\,r(t)$$
$$r(t+1) = -.12\,c(t) + 1.14\,r(t)$$

$$\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix} = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix} \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

explain coefficients qualitatively; is this a reasonable model?

this is an example of a discrete dynamical system

discrete: time changes in discrete units, rather than continuously (derivs rather than difference)

dynamical: changes in time

system: more than 1 fcn being studied

e.g.
$$\frac{dc}{dt} = -.14\,c(t) + .08\,r(t)$$
$$\frac{dr}{dt} = -.12\,c(t) + .14\,r(t).$$

Goal: understand the long-time behavior of the coyote-rabbit system, in terms of initial state

$$\vec{x}(0) = \begin{bmatrix} c(0) \\ r(0) \end{bmatrix} \xrightarrow{A} \vec{x}(1) \xrightarrow{A} \vec{x}(2) \xrightarrow{A} \vec{x}(3) \longrightarrow \dots \rightarrow x(t) \rightarrow \dots$$

$$\vec{x}(t) = A^t \vec{x}_0.$$ e.g. $\vec{x}_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, $\vec{x}(10) = A^{10}\vec{x}_0 \approx \begin{bmatrix} 80 \\ 170 \end{bmatrix}$ (technology?)

$\boxed{\text{Case 1}}$: (special choice of lots of roadrunners relative to coyotes)

$$\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix} \qquad \vec{x}(1) = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix} = (1.1)\begin{bmatrix} 100 \\ 300 \end{bmatrix} = (1.1)\vec{x}_0$$

so
$$\begin{aligned} \vec{x}(2) &= A\,\vec{x}(1) \\ &= A\,(1.1)\vec{x}_0 \\ &= (1.1)\,A\vec{x}_0 \\ &= (1.1)^2\,\vec{x}_0 \end{aligned}$$

$\vec{x}(t) = (1.1)^t\,\vec{x}_0$; $\quad c(t) = (1.1)^t\,100$, $\quad r(t) = (1.1)^t\,300$ $\quad$ exponential growth both populations!

$\boxed{\text{Case 2}}$

$$\frac{\overset{86}{2}}{172}$$

$$\vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix} \quad ; \text{ tougher times.}$$

$$\vec{x}(1) = A\vec{x}_0 = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = .9\,\vec{x}_0$$

$$\begin{aligned} \vec{x}(2) &= A(A\vec{x}_0) \\ &= A(.9\vec{x}_0) \\ &= .9\,A(\vec{x}_0) = (.9)^2\,\vec{x}_0 \end{aligned}$$

$$\vec{x}(t) = .9^t \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$ bad news!

$\boxed{\text{Case 3}}$ $$\vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$$

$$\vec{x}(1) = \begin{bmatrix} .86 & .08 \\ -.12 & 1.14 \end{bmatrix}\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 940 \\ 1020 \end{bmatrix}$$

$$\vec{x}(2) = \begin{bmatrix} 890 \\ 1050 \end{bmatrix}, \quad \vec{x}(3) = \begin{bmatrix} 849.4 \\ 1090.2 \end{bmatrix}$$

$$\vec{x}(4) \approx \begin{bmatrix} 818 \\ 1141 \end{bmatrix}, \quad \vec{x}(5) \approx \begin{bmatrix} 795 \\ 1203 \end{bmatrix}$$

$$\ldots \quad \vec{x}(10) \approx \begin{bmatrix} 798 \\ 1696 \end{bmatrix}$$

... no obvious pattern.

$$\vec{x}(20) \approx \begin{bmatrix} 1443 \\ 4085 \end{bmatrix}$$

The problem is, we're using the wrong basis!

Note $$\underset{\vec{x}_0}{\begin{bmatrix} 1000 \\ 1000 \end{bmatrix}} = 2\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$
$$= 2\vec{v}_1 + 4\vec{v}_2$$

So $$\begin{aligned} A\vec{x}_0 &= A(2\vec{v}_1 + 4\vec{v}_2) \\ &= 2A\vec{v}_1 + 4A\vec{v}_2 \\ &= 2(1.1)\vec{v}_1 + 4(.9)\vec{v}_2 \end{aligned}$$

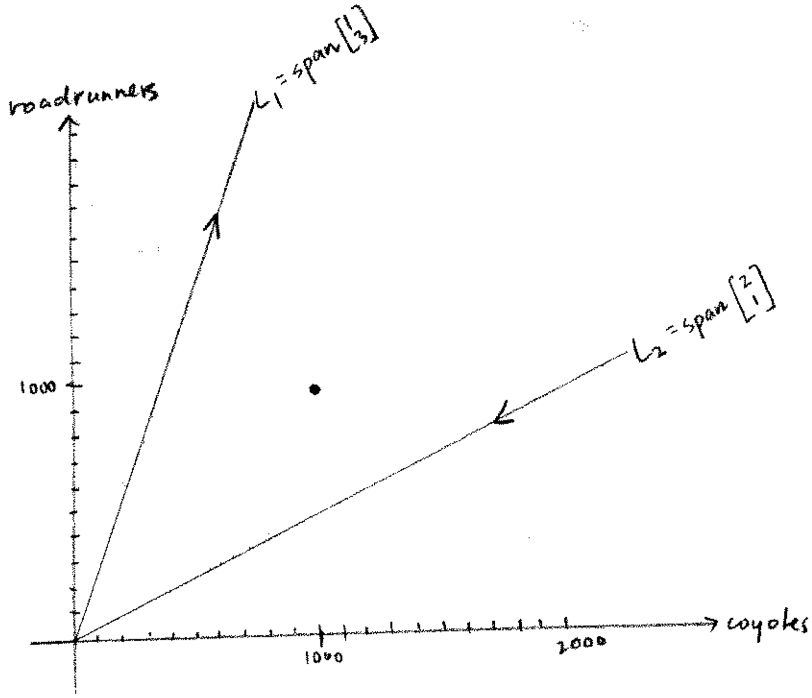
$$\vec{x}(2) = A^2\vec{x}_0 = 2(1.1)^2\vec{v}_1 + 4(.9)^2\vec{v}_2$$

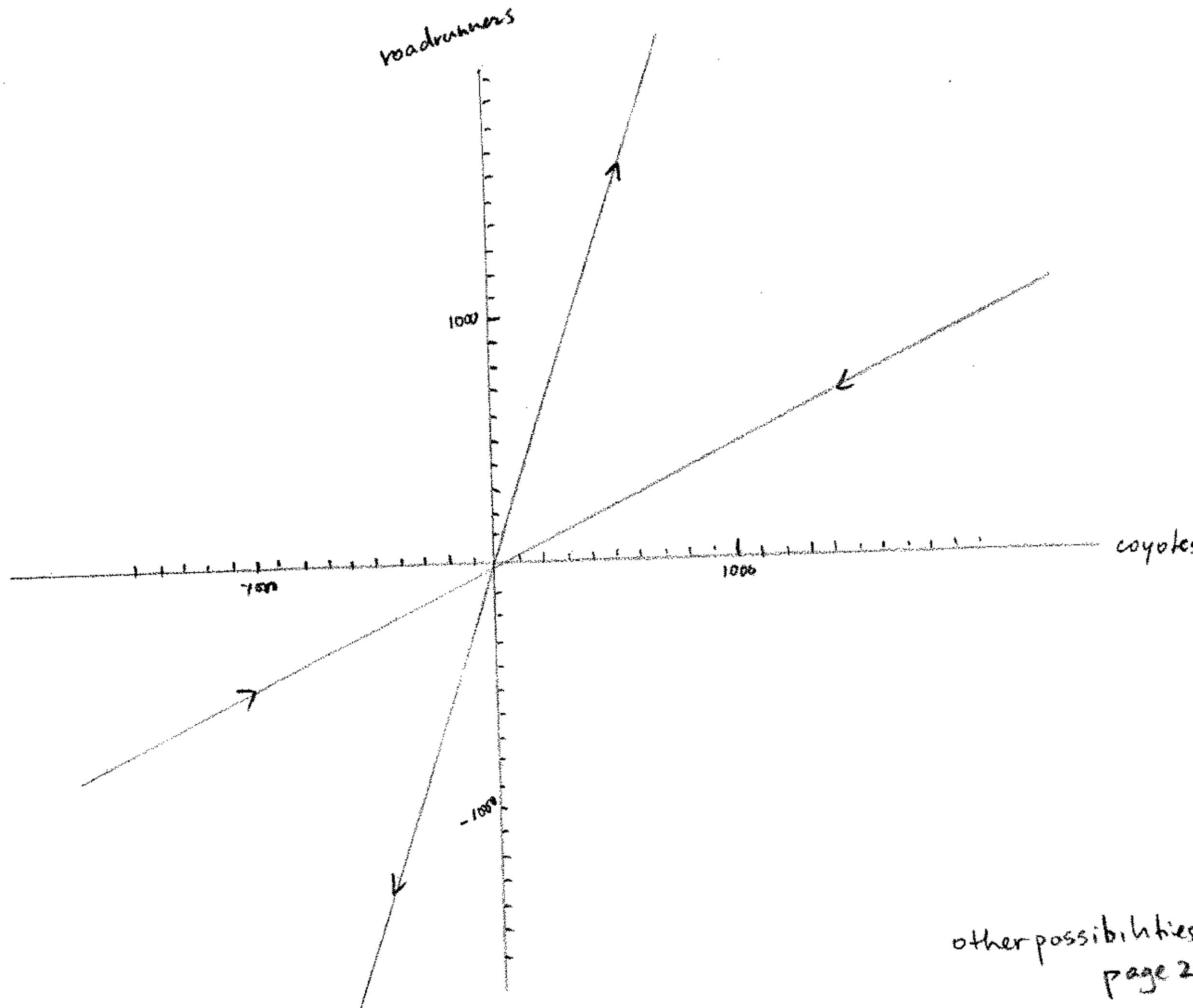
$$\vec{x}(t) = 2(1.1)^t\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(.9)^t\begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$$

<u>Now</u>, you see long time behavior (into the future, and the past)

reproduce the sketches on page 295-296 ...

phase portraits





other possibilities page 297

Big picture:

- $A_{n\times n}$, $\vec{v}$ is an eigenvector of A, with eigenvalue λ
  iff $A\vec{v} = \lambda\vec{v}$

- Application: Consider the discrete dynamical system $\begin{cases} \vec{x}(t+1) = A\,\vec{x}(t), & A_{n\times n} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$

  Then $\vec{x}(t) = A^t\vec{x}_0$.

  If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors,
  $A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\dots$ $A\vec{v}_n = \lambda_n\vec{v}_n$

  If $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

  Then $\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \dots + c_n\lambda_n^t\vec{v}_n$

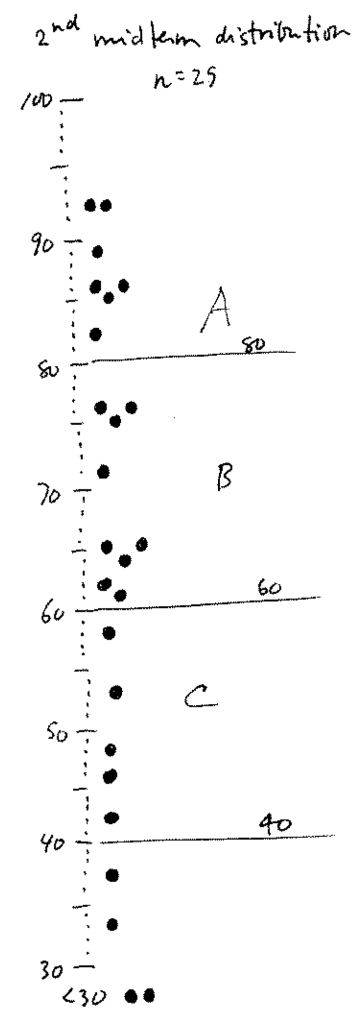
i.e.
$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1^t & & & 0 \\ & \lambda_2^t & & \\ & & \ddots & \\ 0 & & & \lambda_n^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$

$S\vec{c} = \vec{x}_0$

$\vec{c} = S^{-1}\vec{x}_0$

$$x(t) = \underset{P_{E\leftarrow\mathcal{B}}}{S} \; \underset{[L]_{\mathcal{B}}^t}{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^t} \; \underset{P_{\mathcal{B}\leftarrow E}}{S^{-1}} \vec{x}_0$$



eigenvectors and eigenvalues
for some of our favorite geometric transformations:

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (dilations)

rotations

reflections

projections