Math 2270-1
Tuesday Nov 1

- Monday notes on $n \times n$ det def & math induction.

- The "big" theorem is that you can compute the determinant by expanding down any column or across any row. This is a substantial proof by induction (see last 2 pages of today's notes)

<u>example</u>: $\begin{vmatrix} 4 & 12 & 6 & 5 \\ 0 & 0 & 3 & 0 \\ 7 & 0 & 15 & 0 \\ 0 & 0 & 6 & 5 \end{vmatrix}$

<u>Theorem</u> Let $[A]_{n\times n}$. Define

$$T(\vec{x}) = \begin{vmatrix} col_1(A) & \cdots & \underset{\underset{col_j}{\uparrow}}{\vec{x}} & \cdots & col_n(A) \end{vmatrix}$$

(det of A, with j^{th} column replaced by $\vec{x}$).

$$L(\vec{x}) = \begin{vmatrix} row_1(A) \\ \vdots \\ \vec{x}^T \\ \vdots \\ row_n(A) \end{vmatrix}$$

(det of matrix obtained from A by replacing $row_i(A)$ with $\vec{x}^T$).

Then $T, L : \mathbb{R}^n \to \mathbb{R}$ are both linear.

<u>proof</u> (for L): $L(\vec{x}+\vec{y})$ = (expand across row i)

$$= \sum_{j=1}^{n} (-1)^{i+j} (x_j + y_j) |A_{ij}|$$

$$= \sum_{j=1}^{n} (-1)^{i+j} x_j |A_{ij}| + \sum_{j=1}^{n} (-1)^{i+j} y_j |A_{ij}| = L(\vec{x}) + L(\vec{y})$$

$$L(c\vec{x}) = \sum_{j=1}^{n} (-1)^{i+j} (cx_j) |A_{ij}| = c \sum \quad = cL(\vec{x}) \quad ■$$

Theorem:
effect of elementary row operations of determinants. (or col ops)

① swapping two rows changes the sign of the det ; so if two rows are equal, det=0 !

② multiplying a single row by c multiplies a det by c

(so factoring c out of a single row, factors c out of the det).

③ replacing $\text{row}_i(A)$ with $\text{row}_i(A) + c\,\text{row}_k(A)$ $(k \neq i)$

does not change det value!!

proof ① : by induction! $n=2$: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$; $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb-ad$ ✓

if true $\forall$ $(n-1)\times(n-1)$ dets, and $n \geqslant 3$, suppose $\text{row}_i(A)$ & $\text{row}_k(A)$ are swapped. Then expand across row_ℓ, some $\ell \neq i,k$.

$$|A| = \sum_{j=1}^{n} a_{\ell j}(-1)^{\ell+j}\,|A_{\ell j}|$$

$$|A_{swap}| = \sum_{j=1}^{n} a_{\ell j}(-1)^{\ell+j}\,\underbrace{|A_{\ell j\ \text{swapped}}|}_{=-|A_{\ell j}|}$$ by induction! ■

If 2 rows are equal, then swapping yields $|A|$ and $-|A|$

so $|A|=0$!

② Special case of linearity theorem on page 1

③ from linearity:

$$\begin{vmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_i(A) + c\,\text{row}_k(A) \\ \vdots \\ \text{row}_n(A) \end{vmatrix} = |A| + \underbrace{\begin{vmatrix} \text{row}_1(A) \\ \text{row}_k(A) \\ \\ c\,\text{row}_k(A) \\ \vdots \\ \text{row}_n(A) \end{vmatrix}}$$

$$= c \begin{vmatrix} \text{row}_k(A) \\ \vdots \\ \text{row}_k(A) \end{vmatrix} = 0.$$ ■

<u>example</u>

compute $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix}$ using elementary row operations

Theorem : (determinant can be computed by expanding across any row or down any column)

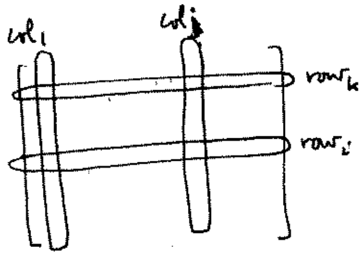
If A is $n\times n$, then

$$\underset{\substack{\uparrow\\ \text{def}}}{|A|} = \sum_{i=1}^{n}(-1)^{i+1}a_{i1}\,|A_{i1}| \overset{①}{=} \underset{\substack{\text{any } 1\le j\le n\\ \text{exp. down col}_j}}{\sum_{i=1}^{n}(-1)^{i+j}a_{ij}\,|A_{ij}|} \overset{②}{=} \underset{\substack{\text{any } 1\le i\le n\\ \text{exp. across row } i}}{\sum_{j=1}^{n}(-1)^{i+j}a_{ij}\,|A_{ij}|}$$

proof is by induction.
we've already verified $n=2,3$. (or, you should have, anyway).

* So we assume theorem is true for $(n-1)\times(n-1)$ matrices, and show this implies it for $n\times n$ matrices.

① $|A| := \sum_{i=1}^{n}(-1)^{i+1}a_{i1}\,|A_{i1}|$ Does this equal $\sum_{k=1}^{n}(-1)^{k+j}a_{kj}\,|A_{kj}|$?



expand $|A_{i1}|$ down col $j-1$ (using induction hyp!)

$$|A_{i1}| = \sum_{k=1}^{i-1}(-1)^{k+j-1}a_{kj}\,|A_{i1,kj}| + \sum_{k=i+1}^{n}(-1)^{k-1+j-1}a_{kj}\,|A_{i1,kj}|$$

← deleted rows i,k cols $1,j$

expand $|A_{kj}|$ down col$_1$: (using induction hyp!)

$$|A_{kj}| = \sum_{i<k}a_{i1}(-1)^{i+1}|A_{i1,kj}| + \sum_{i>k}a_{i1}(-1)^{i-1+1}|A_{i1,kj}|$$

Thus

col$_j$ expansion

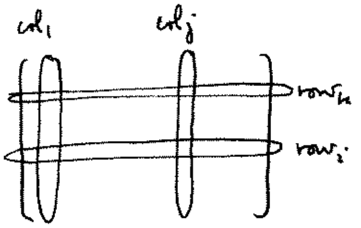
Thus

$$\begin{tikzcd}
{|A| = \sum_{k<i}(-1)^{i+1+k+j-1}a_{i1}a_{kj}\,|A_{i1,kj}|} \arrow[dr, leftrightarrow, "="] & {= \sum_{i<k}(-1)^{k+j+i+1}a_{kj}a_{i1}\,|A_{i1,kj}|} \\
{+ \sum_{k>i}(-1)^{i+1+k-1+j-1}a_{i1}a_{kj}\,|A_{i1,kj}|} \arrow[ur, leftrightarrow, "="] & {+ \sum_{i>k}(-1)^{k+j+i}a_{kj}a_{i1}\,|A_{i1,kj}|}
\end{tikzcd}$$

∎

② $|A| := \sum_{k=1}^{n} (-1)^{k+1} a_{k1} |A_{k1}|$ Does this equal $\sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$?

row$_i$ expansion



for $j>1$, expand down 1st col

$$|A_{ij}| = \sum_{k=1}^{i-1} (-1)^{1+k} a_{k1} |A_{ij,k1}| + \sum_{k=i+1}^{n} (-1)^{1+k-1} a_{k1} |A_{ij,k1}|$$

$k<i$ $|A_{k1}|$ across $i-1^{st}$ row

$$= \sum_{j=2}^{n} a_{ij} (-1)^{i-1+j-1} |A_{k1,ij}|$$

$k>i$ $|A_{k1}|$ across i^{th} row

$$= \sum_{j=2}^{n} a_{ij} (-1)^{i+j-1} |A_{k1,ij}|$$

So row$_i$ expansion =

$j=1$: $(-1)^{i+1} a_{i1} |A_{i1}|$

$$+ \sum_{\substack{k<i \\ j=2 \cdots n}} (-1)^{i+j+k+1} a_{ij} a_{k1} |A_{ij,k1}|$$

$$+ \sum_{\substack{k>i \\ j=2 \cdots n}} (-1)^{i+j+k} a_{ij} a_{k1} |A_{ij,k1}|$$

$\boxed{\text{thus}}$

$$|A| = \sum_{k<i,\ j=2 \cdots n} (-1)^{k+i+j-1} a_{k1} a_{ij} |A_{k1,ij}|$$

$k=i \rightarrow$ $+ (-1)^{i+1} a_{i1} |A_{i1}|$

$$+ \sum_{k>i,\ j=2 \cdots n} (-1)^{k+i+j} a_{k1} a_{ij} |A_{k1,ij}|$$

∎

[redacted]