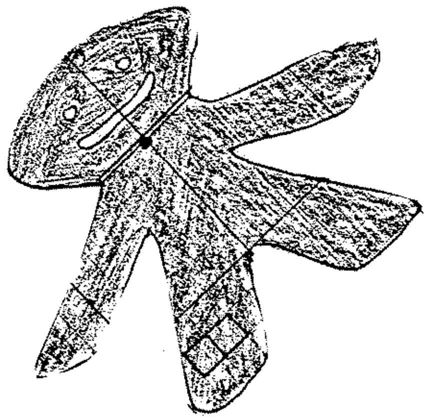
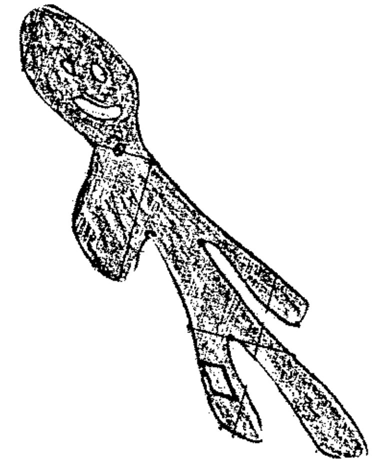
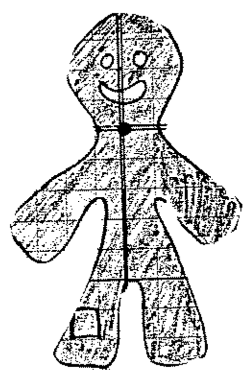


SCALING, SELF SIMILARITY & FRACTALS



2270-1
Fall 2005



Bob

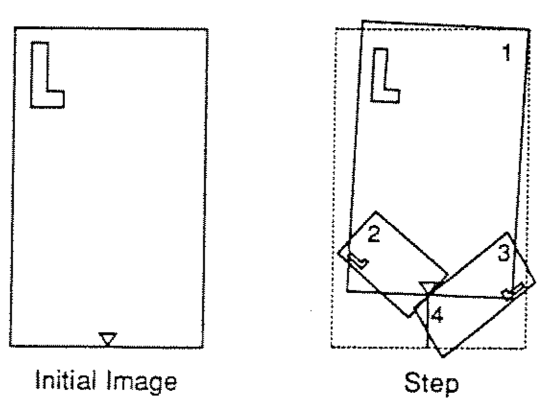


Figure 5.22 : Blueprint of Barnsley's fern.

Figure 5.25 : Barnsley's fern generated by an MRCM with only four lens systems.

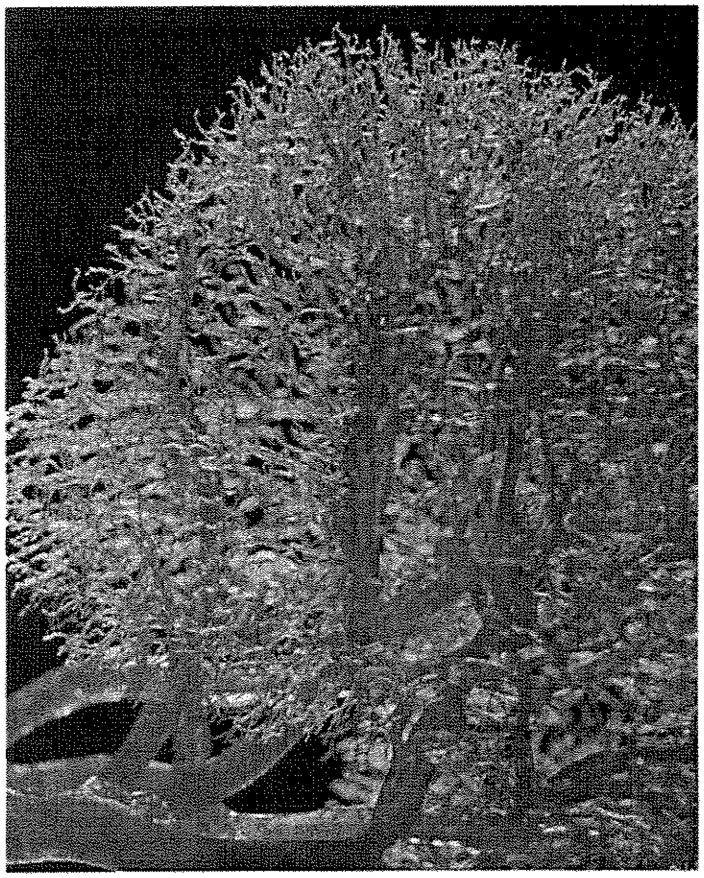


Plate 2: Cast of a child's kidney, venous and arterial system, © Manfred Kage, Institut für wissenschaftliche Fotografie.

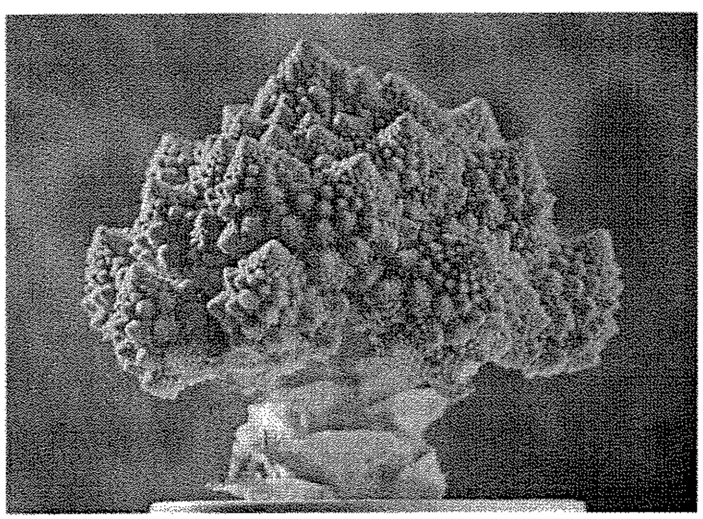


Plate 3: Broccoli Romanesco.

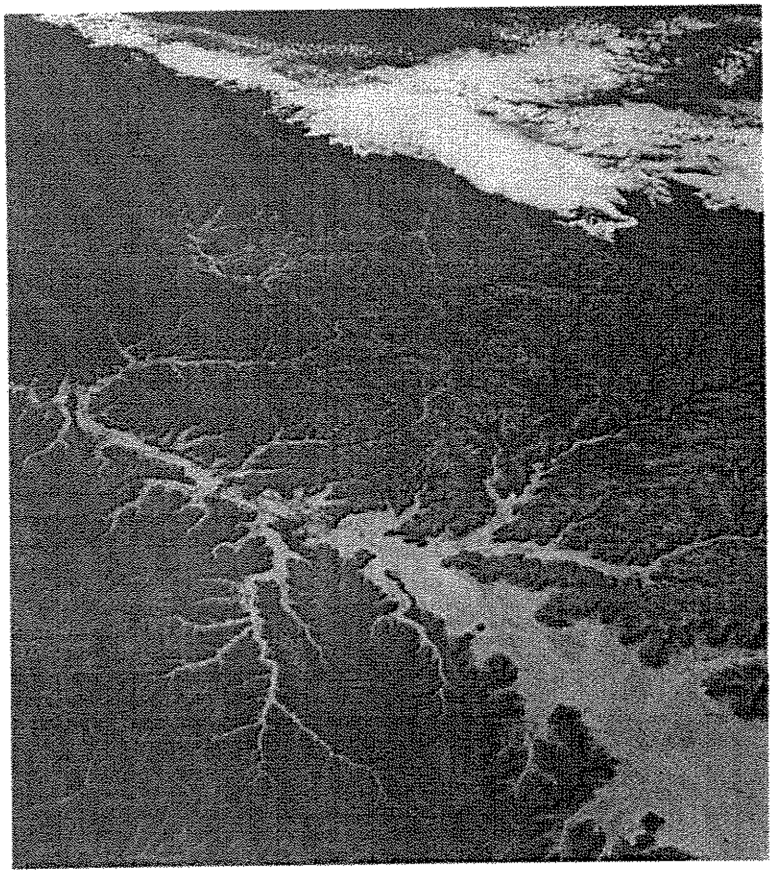


Plate 4: Wadi Hadramaut, Gemini IV image, © Dr. Vehrenberg KG.

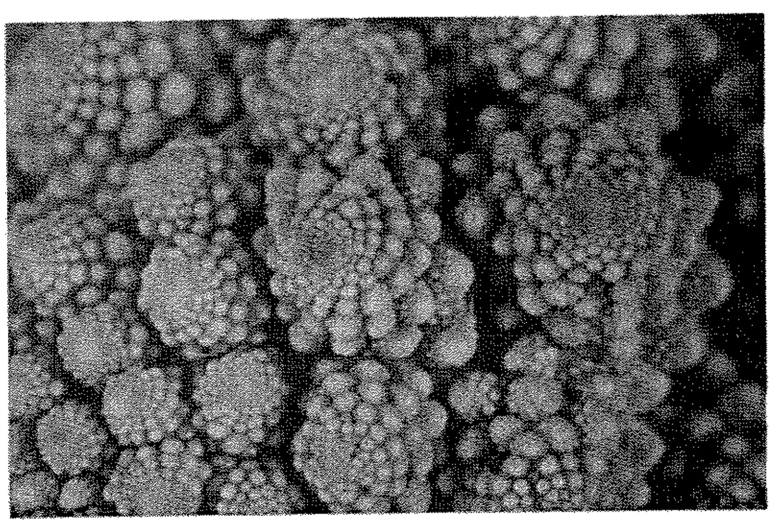


Plate 5: Broccoli Romanesco, detail.

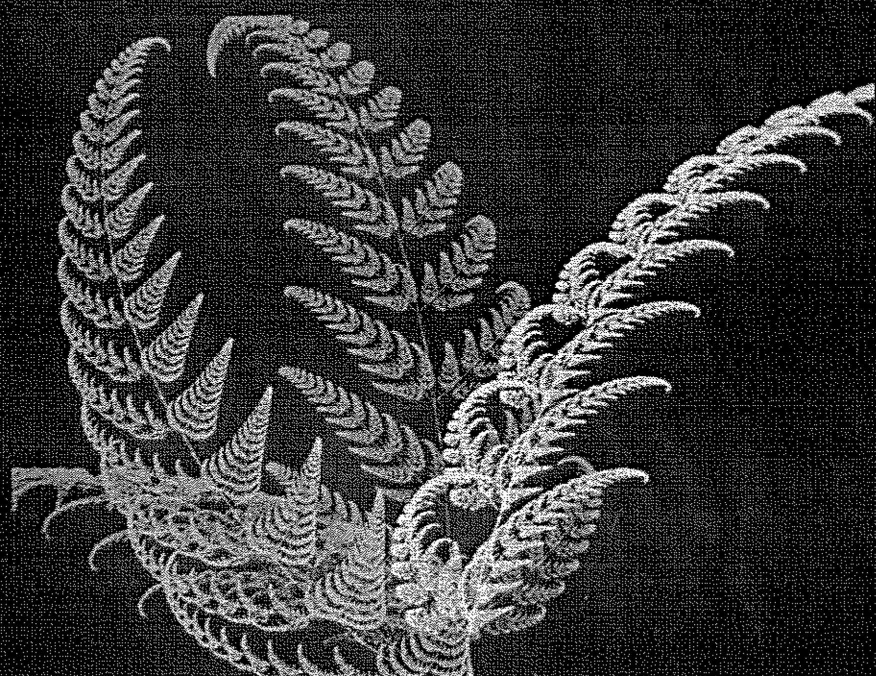


Plate 3.10.1
Three dimensional ferns.

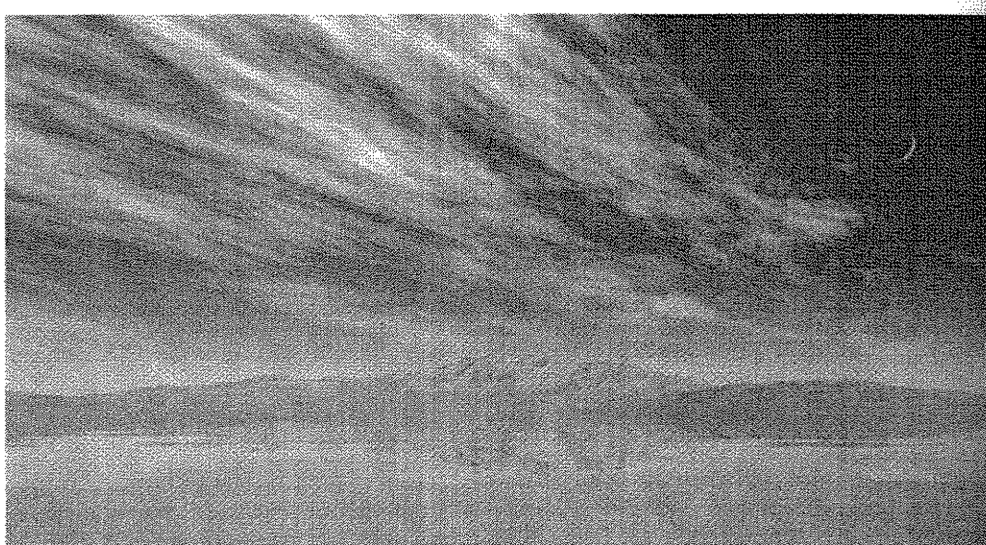


Plate 10: "Zabriski Point", fractal forgery of a mirage, © K. Musgrave, C. Kolb, B.B. Mandelbrot.

Plate 11: "Carolina", fractal forgery, © K. Musgrave.

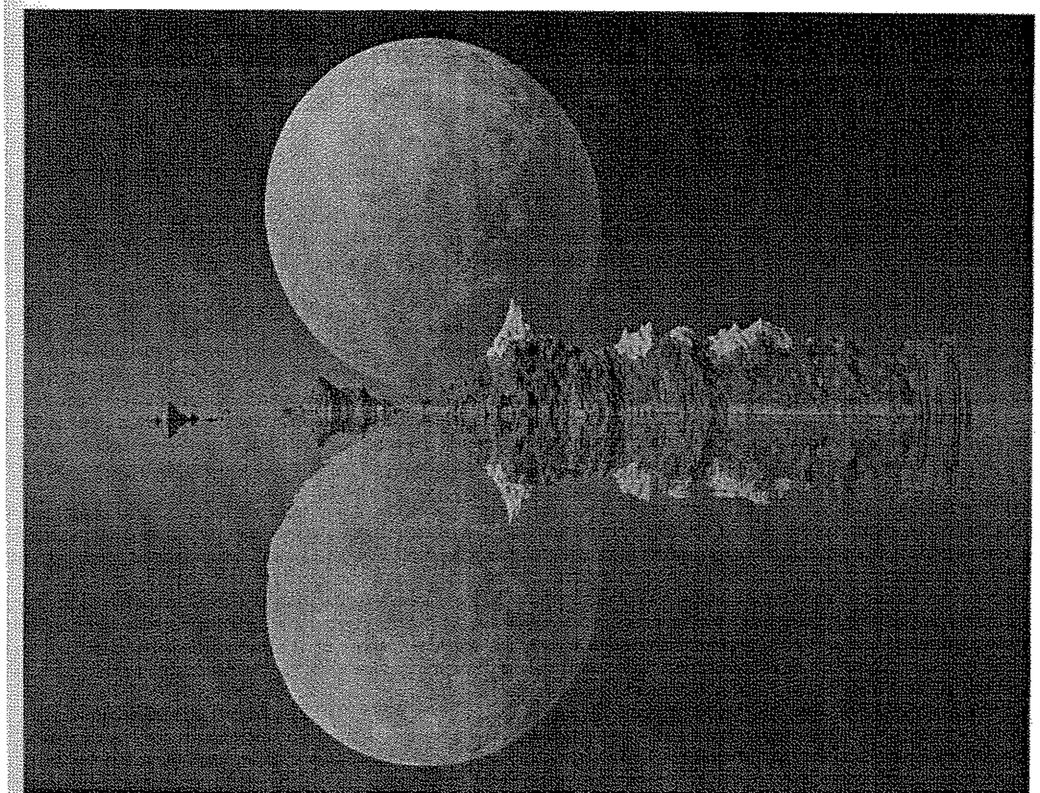


Plate 12: Fractal forgery of planet rise, © K. Musgrave.

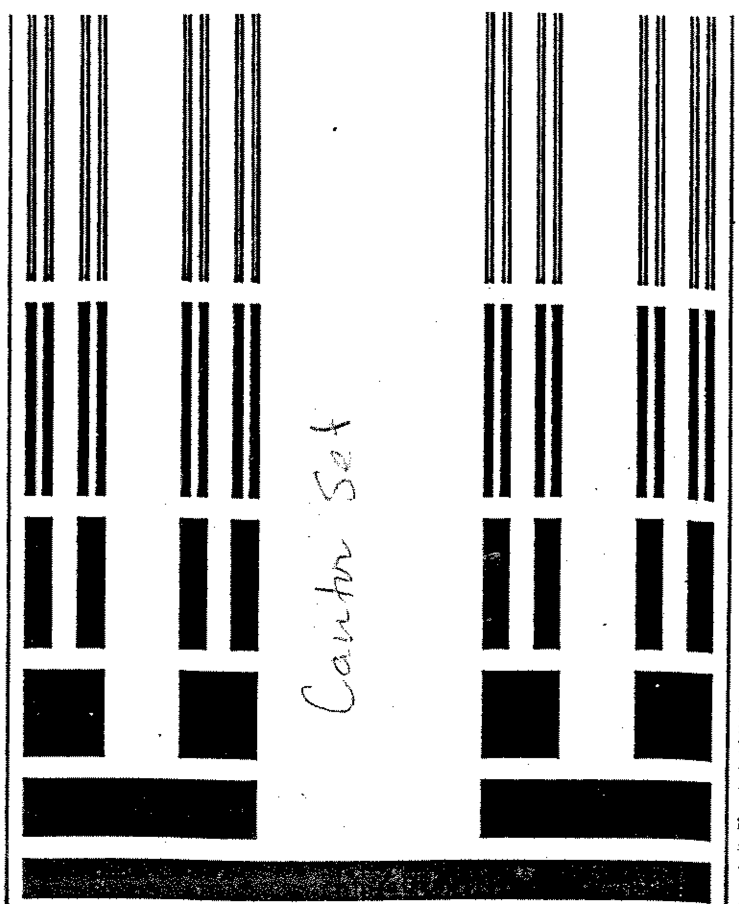
Plate 13: "Ein kleines Nachtlicht", fractal forgery, stereoscopic image. View the left image with your right eye and the right image with your left eye. © K. Musgrave, C. Kolb, B.B. Mandelbrot.

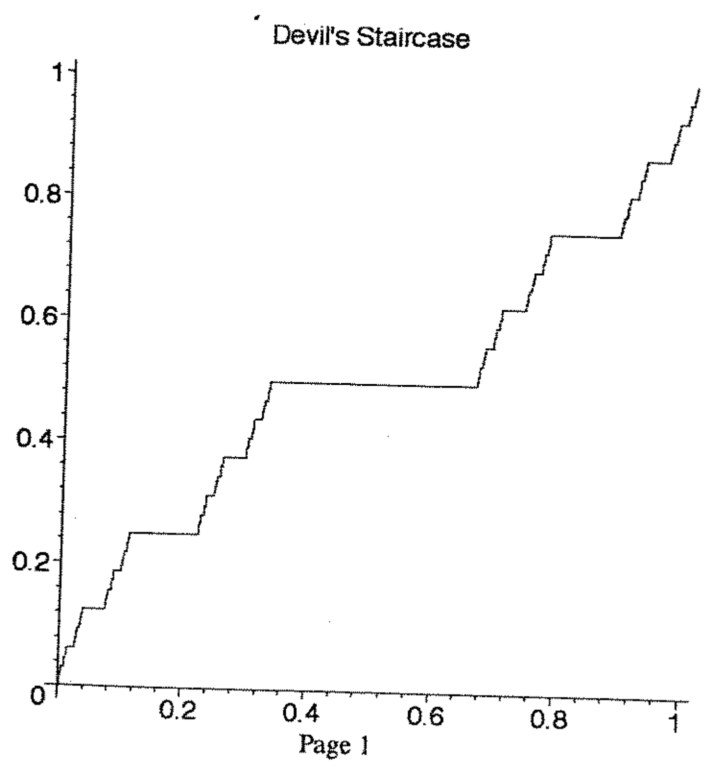
Georg Ferdinand Ludwig Philipp Cantor

2.1

PART II
FRACTALS

Born: 3 March 1845 in St Petersburg, Russia
Died: 6 Jan 1918 in Halle, Germany





Niels Fabian Helge von Koch

Born: 25 Jan 1870 in Stockholm, Sweden
Died: 11 March 1924 in Stockholm, Sweden

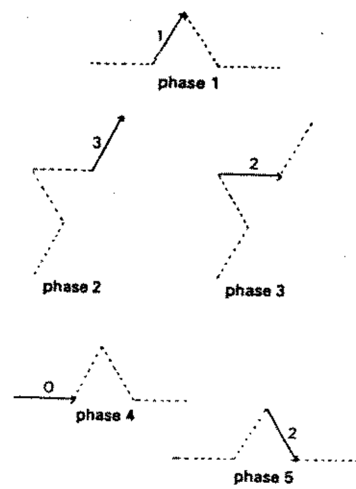
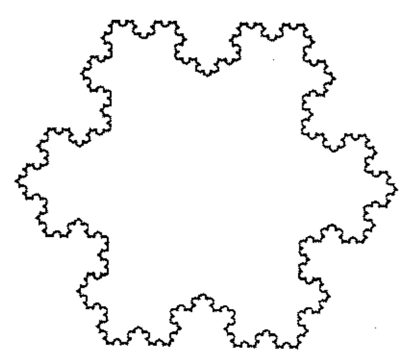


Figure 3.5 Koch fractal and the quaternary system

how long is it?

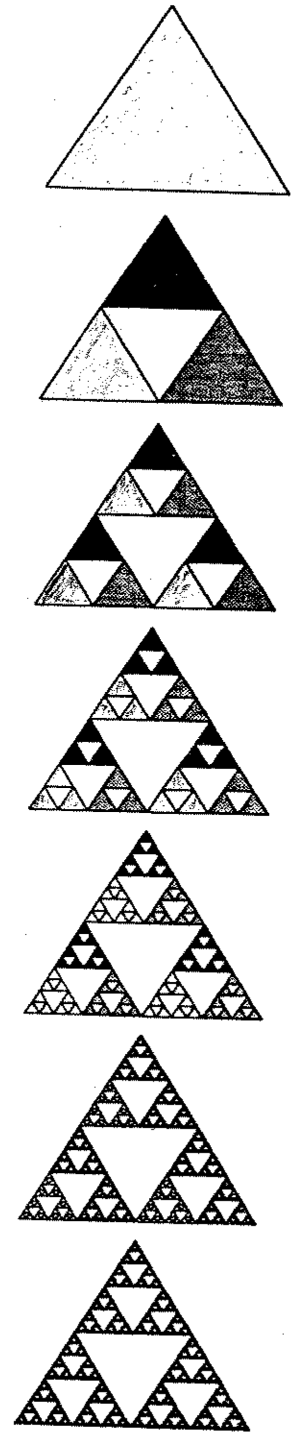


KOCH SNOWFLAKE

2.3

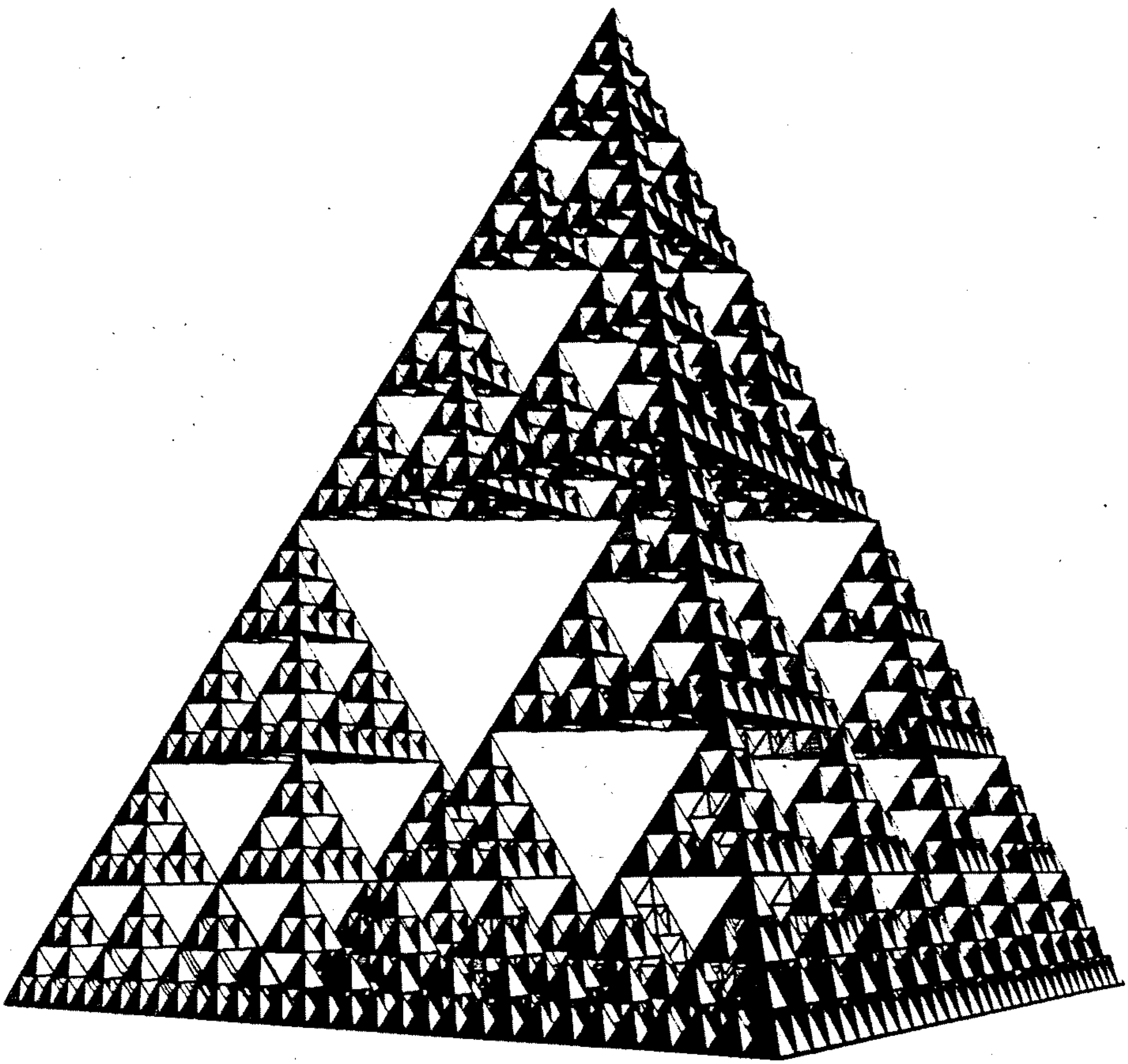
Waclaw Sierpinski

Born: 14 March 1882 in Warsaw, Poland
Died: 21 Oct 1969 in Warsaw, Poland



how much area?

SIERPINSKI TRIANGLE

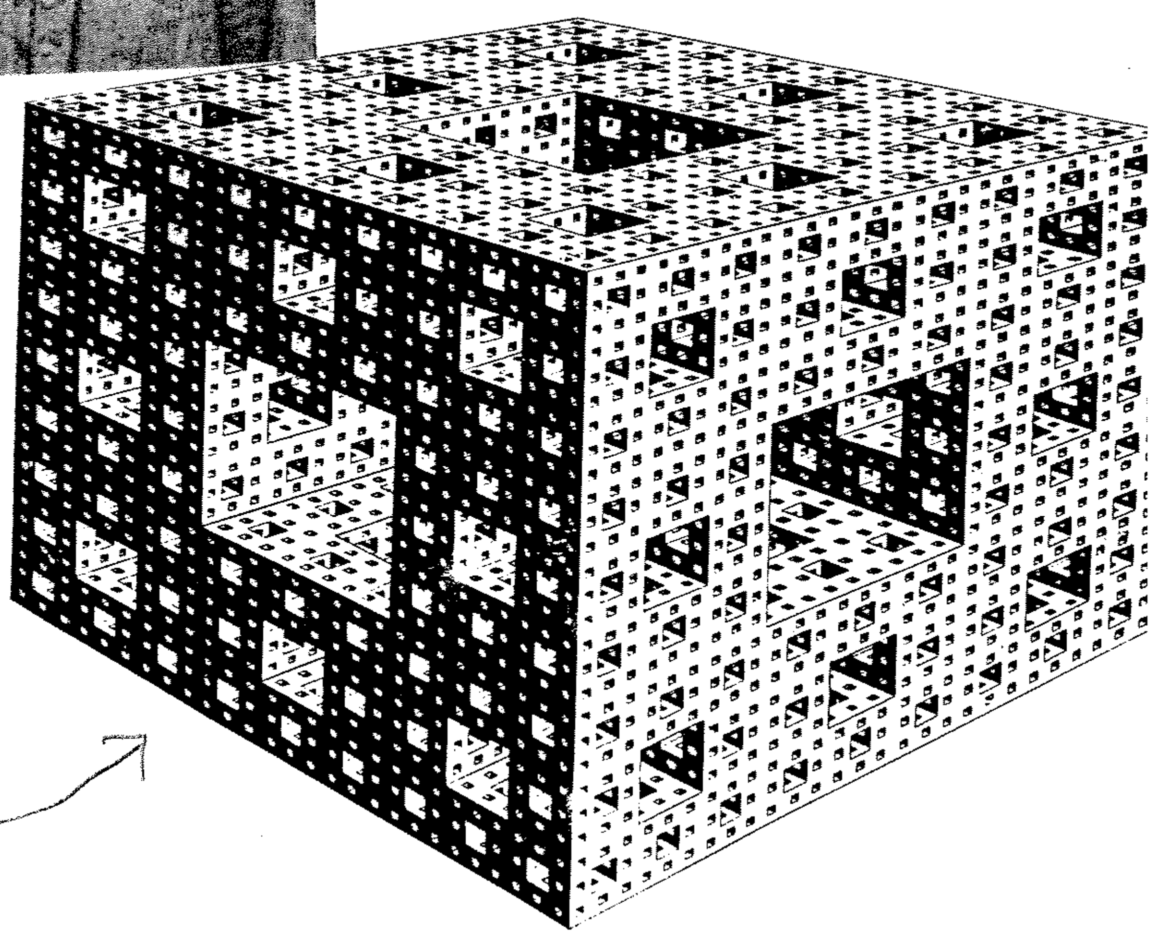


SIERPINSKI PYRAMID

2.5

Karl Menger

Born: 13 Jan 1902 in Vienna, Austria
Died: 5 Oct 1985 in Chicago, Illinois, USA



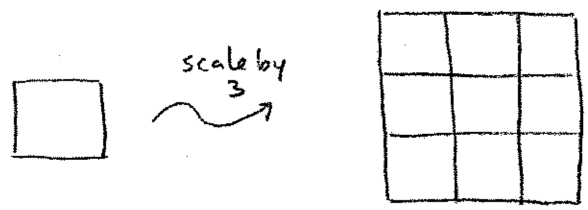
how much volume?

MENGER SPONGE

Scaling, or Self-similarity dimension

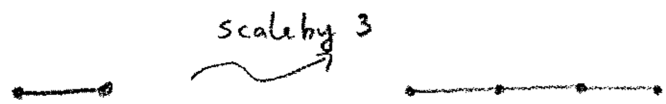
[there is also a "covering" or "Hausdorff" dimension] which is more general

example: If you scale a square by a factor of M, you get M^2 equivalent pieces



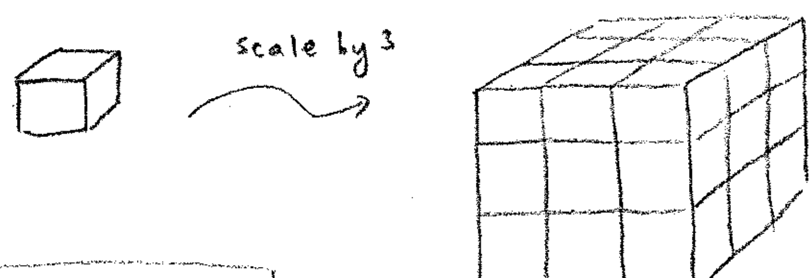
get 3^2 pieces

If you scale a segment by a factor of M, you get M^1 equiv pieces



get 3^1 pieces

If you scale a cube by a factor of M, you get M^3 equiv. pieces

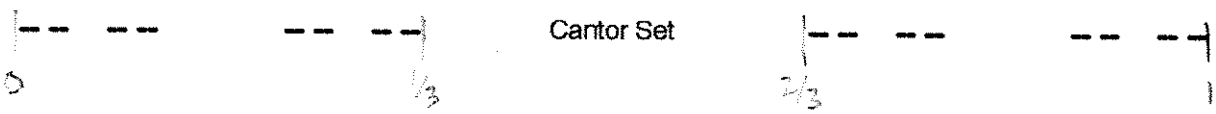


Scaling dimension:

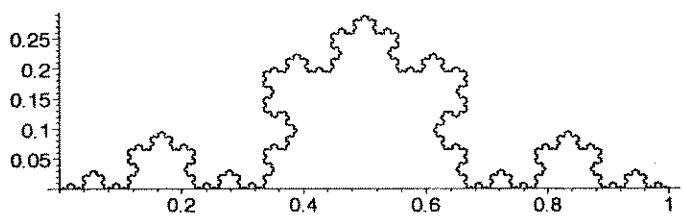
If you scale object by a factor of M and get $M^d = N$ congruent pieces, the scaling dimension of object $:= d$

$$M^d = N$$
$$d \ln M = \ln N$$
$$\boxed{d = \frac{\ln N}{\ln M}}$$

Fractal scaling dimensions:

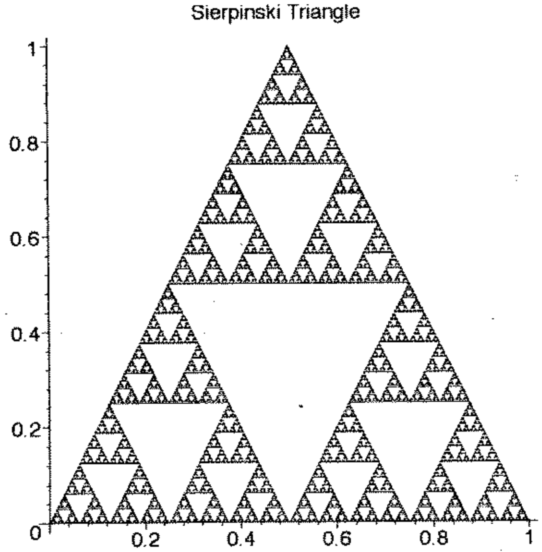


Koch Snowflake



What "dimensions" are these objects?

Sierpinski Triangle



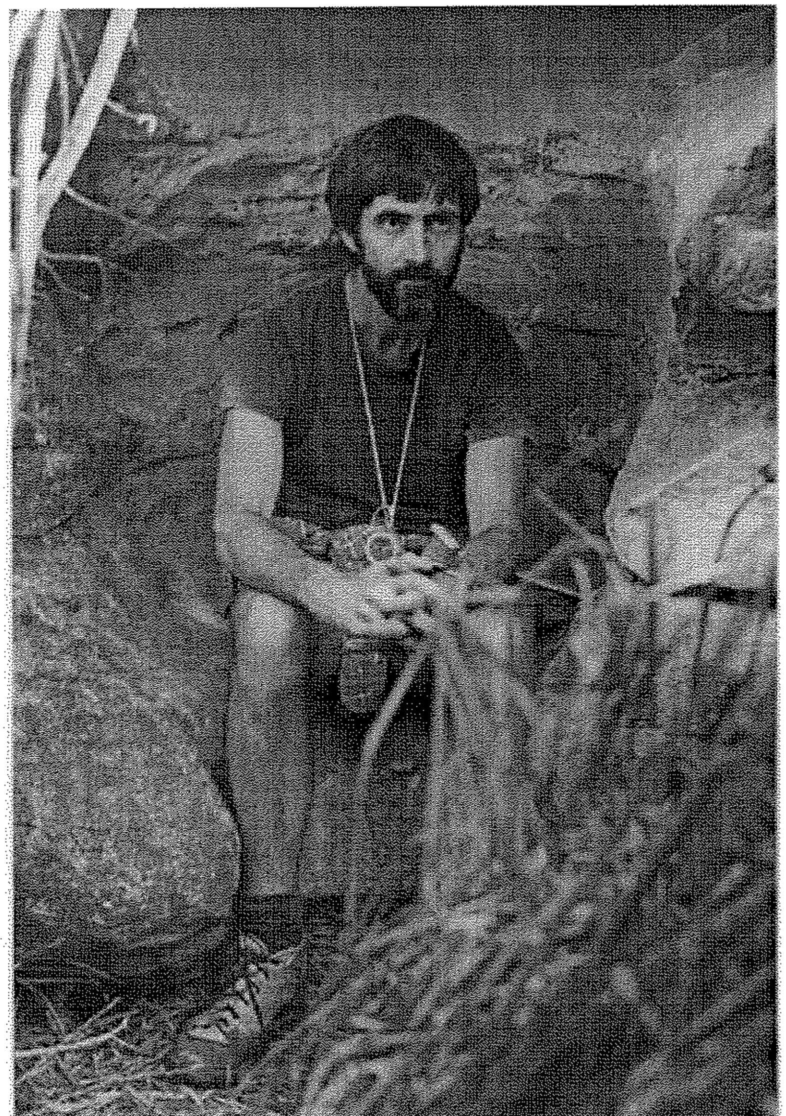
PART III : REVOLUTION

How should one think of fractal construction in general?
Is it really "deletion"?
(Cantor, Koch, etc. were "wrong"!)

→ This new way of thinking is due to John Hutchinson

"Fractals and self-similarity"
Indiana University Math Journal
30 (1981), 713-747

(3.2)

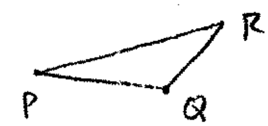


John Hutchinson "Fractals and self-similarity"
Indiana University Math Journal 30 (1981), 713-747

Ingredient 1 : Contractions

Let X be a space on which you can measure distance

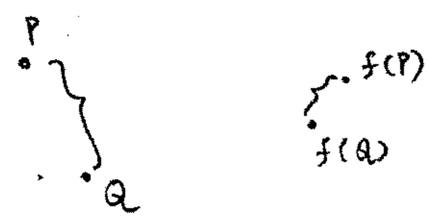
$$\begin{cases} \text{dist}(P,Q) \geq 0 ; \quad = 0 \text{ only when } P = Q \\ \text{dist}(P,Q) = \text{dist}(Q,P) \\ d(P,R) \leq d(P,Q) + d(Q,R) \end{cases}$$



X is called a metric spa

A transformation f of X is
a contraction if there is a fraction μ, $0 \leq \mu < 1$ so that

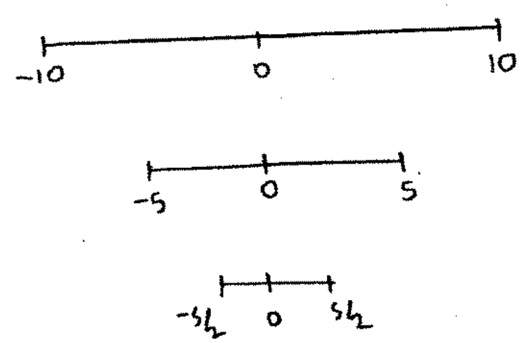
$$d(f(P), f(Q)) \leq \mu \, d(P,Q)$$



Example $f: [-10,10] \to \mathbb{R}$

$f(x) = \frac{1}{2}x$

Is f a contraction?



$= I$

$f(I)$

$f(f(I))$

Which x's satisfy
$f(x) = x$?
these are called fixed points

If we pick any x_0 and set $x_1 = f(x_0)$
$x_2 = f(x_1)$
$x_3 = f(x_2)$
$\vdots$

What happens to x_n as $n \to \infty$?

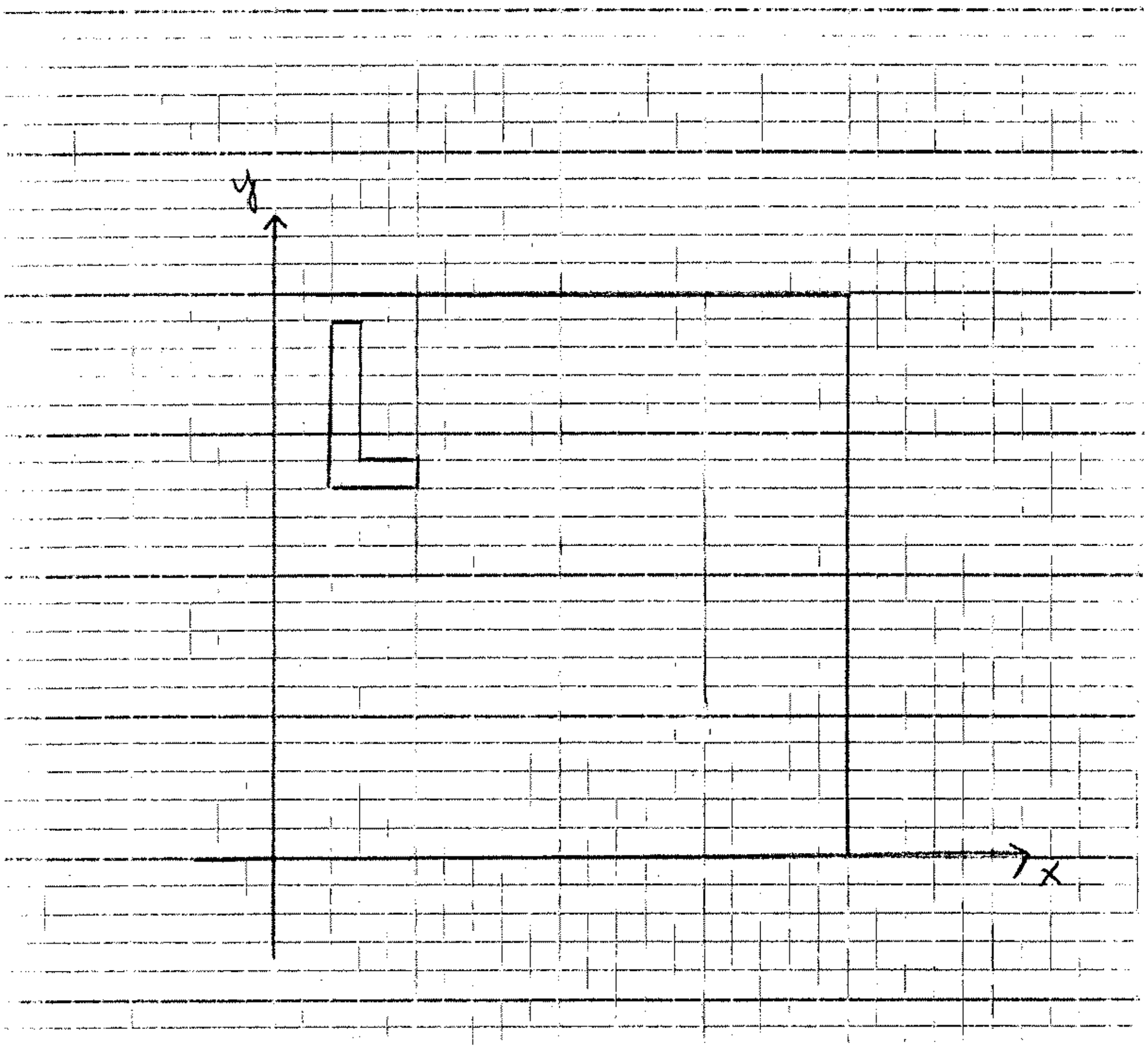
<u>Example</u> $\mathcal{A}$: L-box → L-box

$$\mathcal{A}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{4} \\ \frac{1}{2}y + \frac{1}{4} \end{bmatrix}$$

Is $\mathcal{A}$ a contraction?

Does it have a fixed point?

If we start anywhere, and iterate, do we get to the fixed point?



Banach Fixed Point Principle
(Contraction Mapping Theorem)

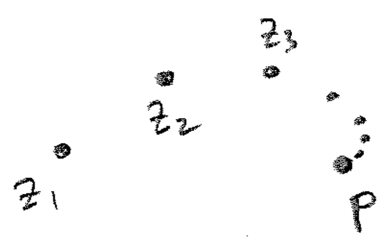
- X a complete metric space
- $f: X \to X$ a contraction mapping

<u>Conclusion</u> f has a unique fixed point P in X, i.e. $f(P) = P$

And for any z_0 in X the iteration

$$z_{n+1} = f(z_n) \qquad n = 0, 1, 2, \dots$$

yields a sequence so that $d(z_n, P) \longrightarrow 0$ as $n \to \infty$



3.6

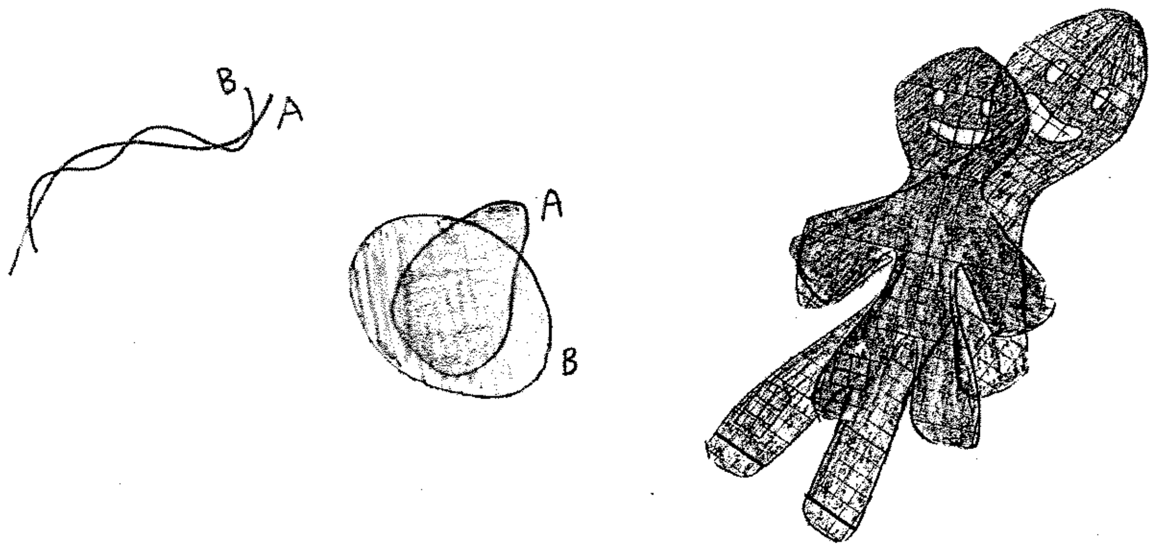
Stefan Banach

Born: 30 March 1892 in Kraków, Austria-Hungary (now Poland)
Died: 31 Aug 1945 in Lvov, Ukraine

Ingrediant 2: Hausdorff distance

Bob, fractals, are all (closed and bounded) subsets of $\mathbb{R}^2$

Hausdorff figured out a good way to measure distance on the space of these subsets

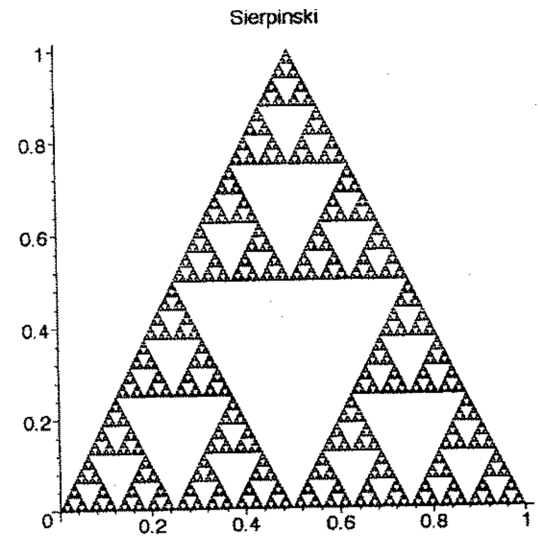


dist(A, B) = the larger of { the maximum distance from points in A to (the nearest point) in B , the max dist from points in B to (their nearest) point in A }

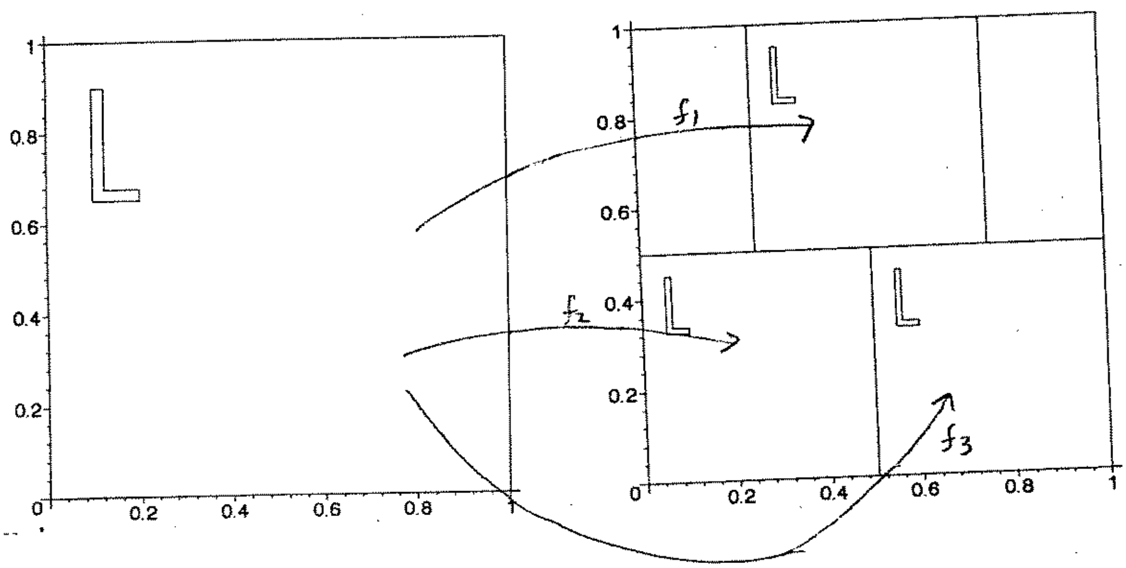
3.8

Felix Hausdorff

Born: 8 Nov 1869 in Breslau, Germany (now Wroclaw, Poland)
Died: 26 Jan 1942 in Bonn, Germany



What Hutchinson saw.



Let S be a set

$$F(S) := f_1(S) \cup f_2(S) \cup f_3(S)$$

e.g. $f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}$

- F is a contraction, using Hausdorff distance.
- Sierpinski Δ is its fixed point!

$S \quad F(S) \quad F(F(S)) \quad F(F(F(S))) \cdots$

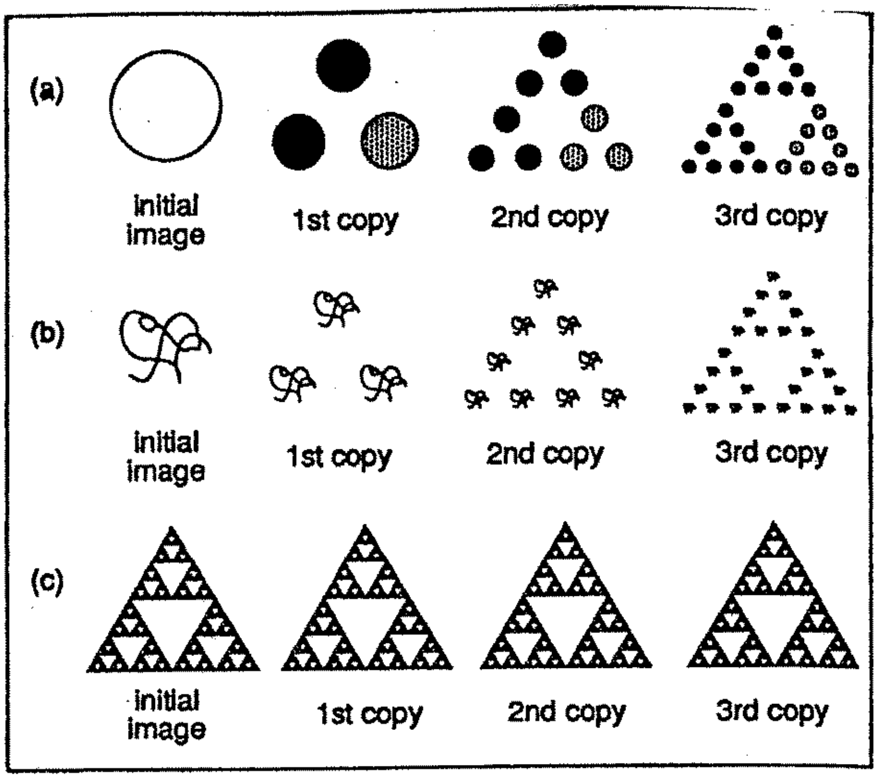


Figure 5.1 : Three iterations of an MRCM with three different initial images.

If we started with Bob he would turn into Sierpinski!

Hutchinson's Theorem 1981

- X a complete metric space

  $f_i : X \to X$ contractions with contraction constants k_i

Let F : the space of subsets of X (the "Bobs") $\longrightarrow$ the space of subsets of X

a metric space with the Hausdorff distance

where $F(\text{Bob}) := f_1(\text{Bob}) \cup f_2(\text{Bob}) \cup \dots \cup f_n(\text{Bob})$

Then F is a contraction, with contraction constant $k = \max(k_1, k_2 \dots k_n)$

- So F has a unique fixed point (our fractal!) and we can get it by picking any initial subset and iterating F!

[First identify desired fractal as such a fixed point!]

3.12

Sierpinski Gasket Variation

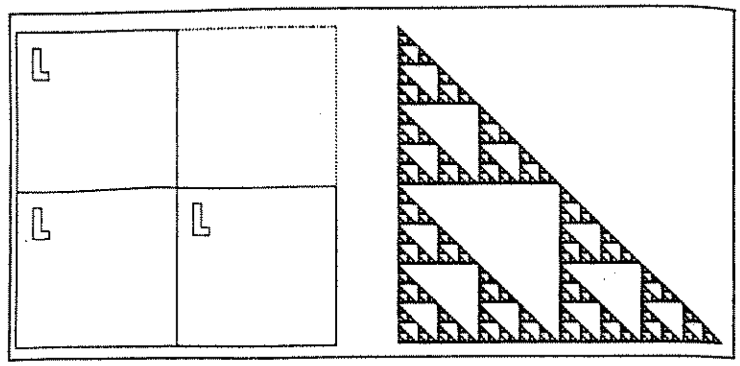


Figure 5.9 : IFS with three similarity transformations with scaling factor 1/2.

The Twin Christmas Tree

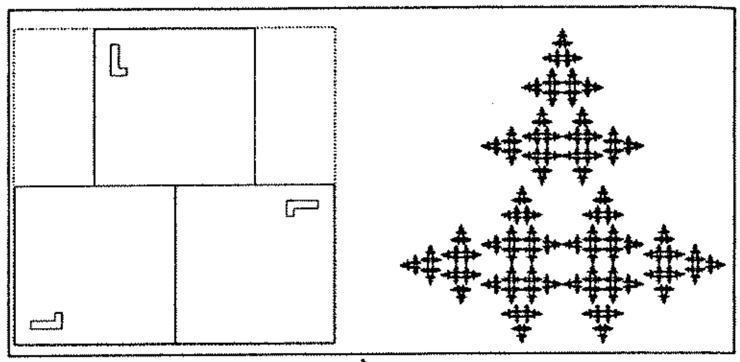


Figure 5.10 : Another IFS with three similarity transformations with scaling factor 1/2.

A Dragon With Threefold Symmetry

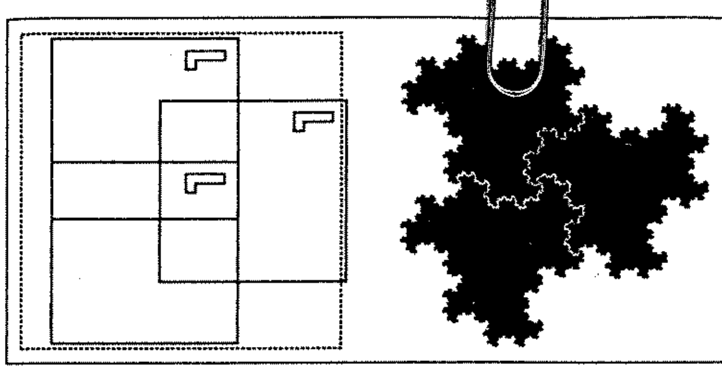


Figure 5.11 : The white line is inserted only to show that the figure can be made up from three parts similar to the whole.

The Cantor Maze

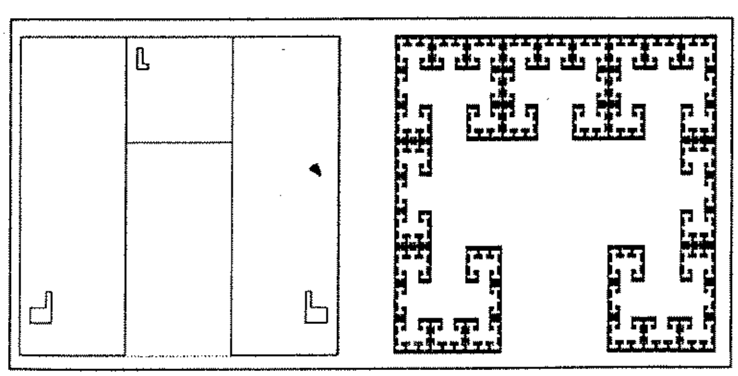


Figure 5.12 : IFS with three transformations, one of which is a similarity. The attrcator is related to the Cantor set.

3.13

IFS for a Twig

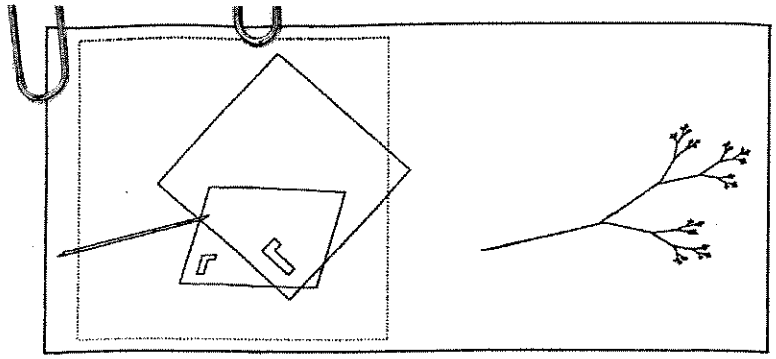


Figure 5.13 : IFS with three affine transformations (no similarities).

Crystal with Four Transformations

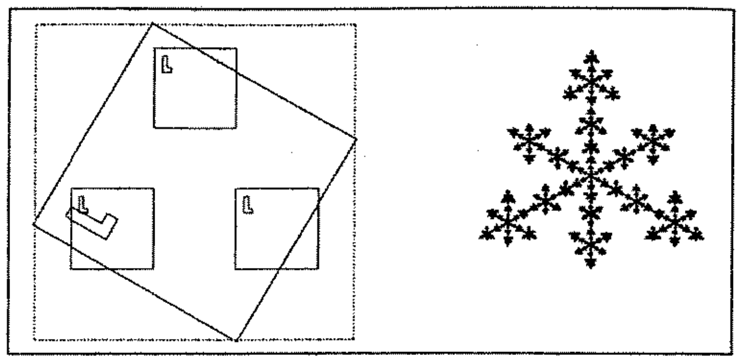


Figure 5.14 : IFS with four similarity transformations.

Triangle, Square, and Circle

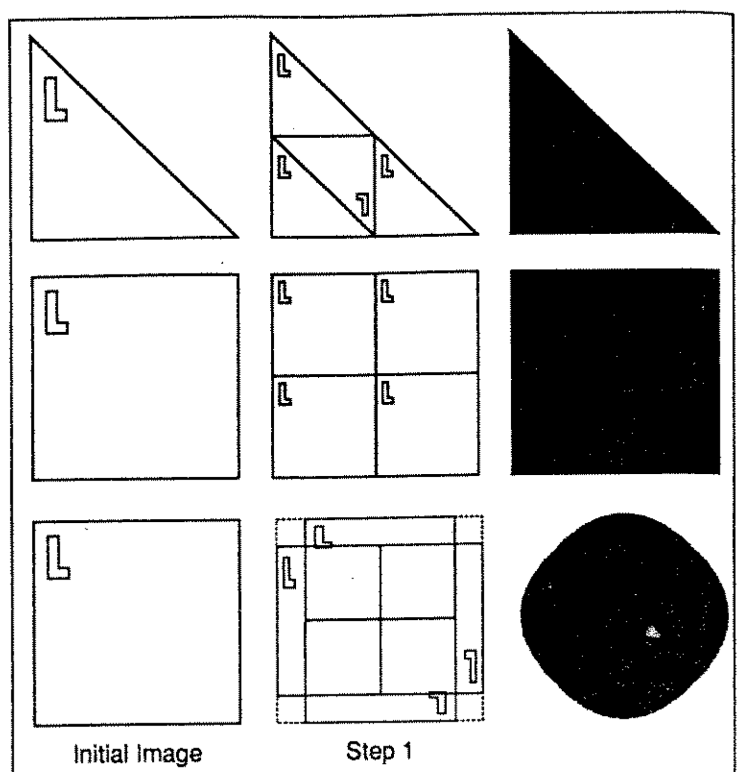


Figure 5.17 : The encoding of a triangle, a square and a circle by IFSs.

3.14

Crystal with Five Transformations

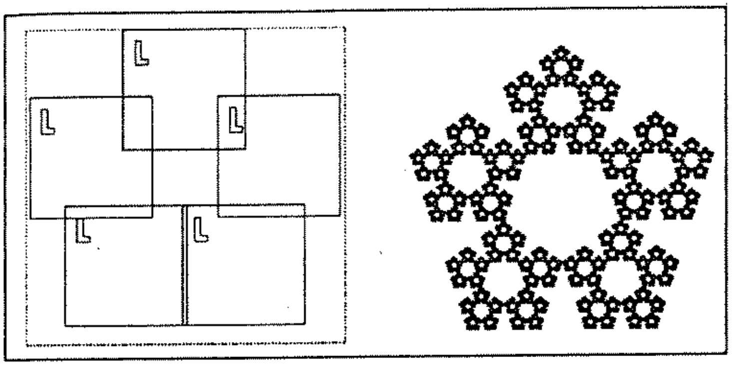
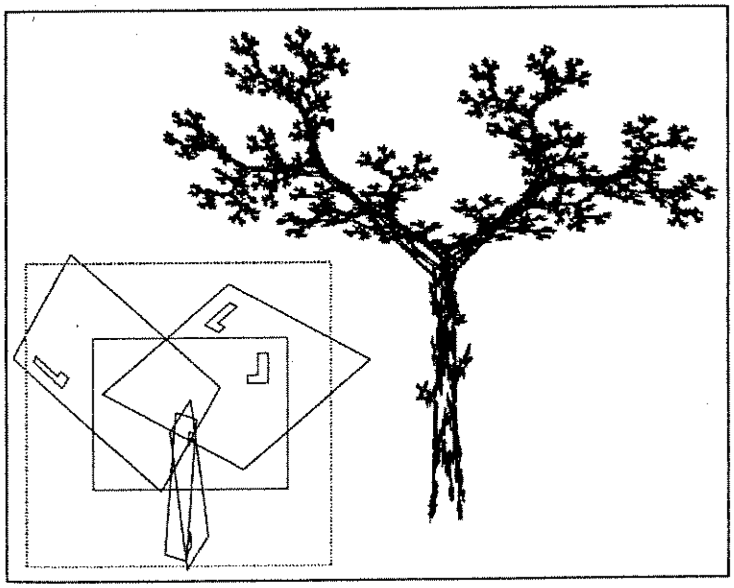
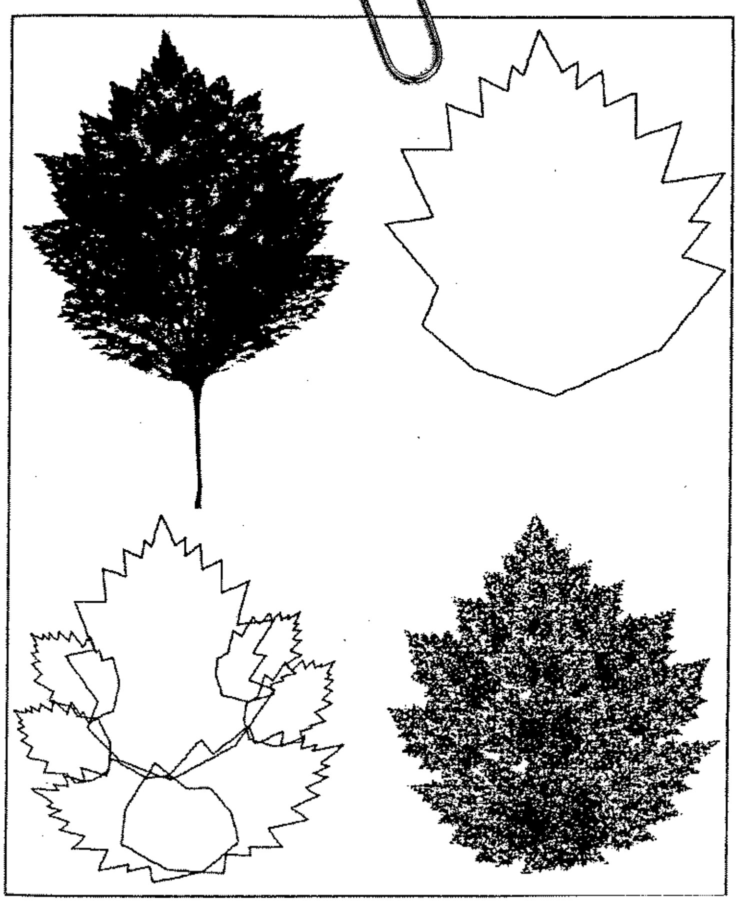


Figure 5.15 : IFS with five similarity transformations. Can you see Koch curves in the attractor?

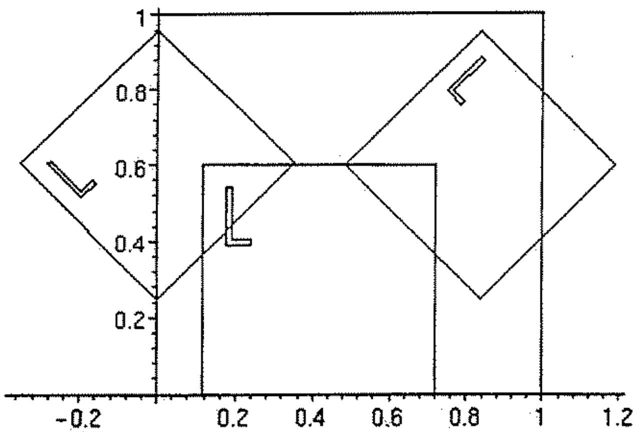
A Tree



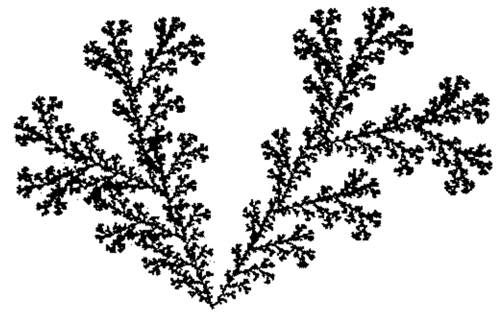
Leaf Collage



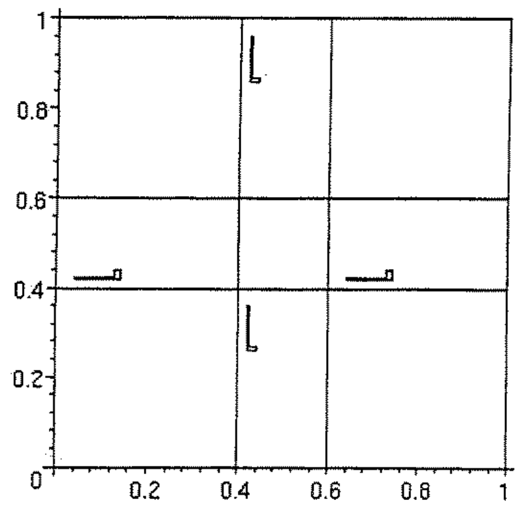
fractal template



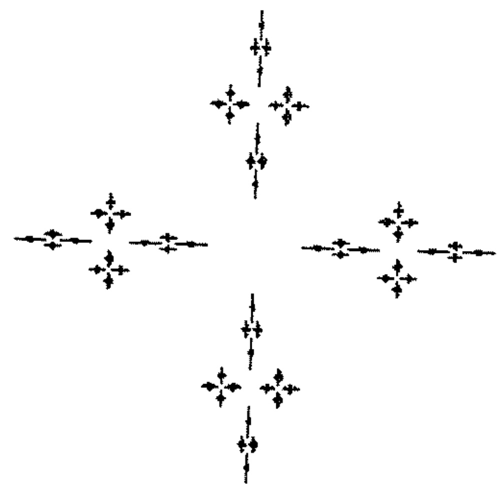
algae



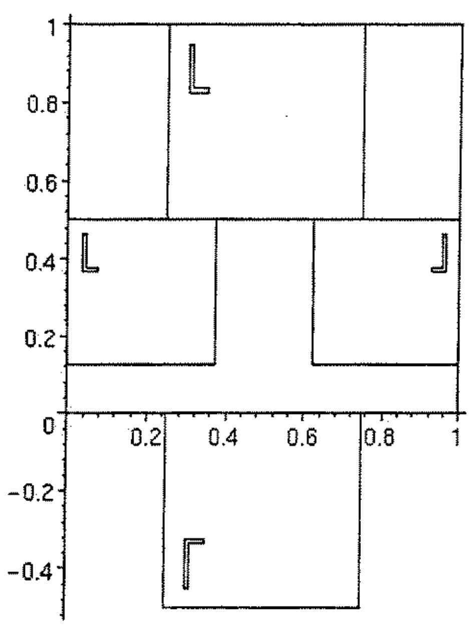
fractal template



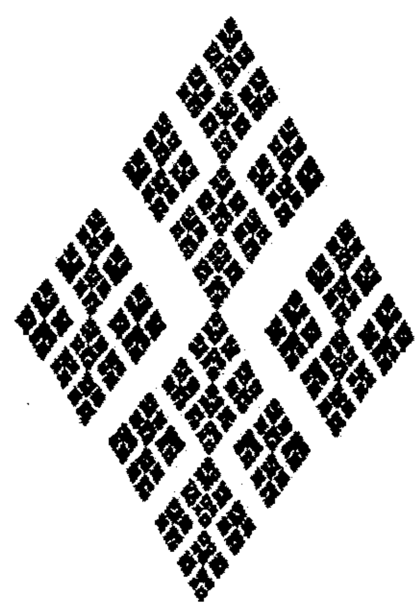
cross-type example



fractal template

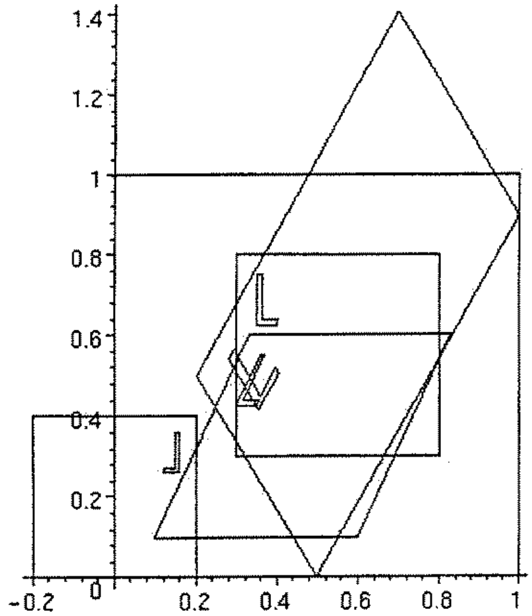


diamond pattern



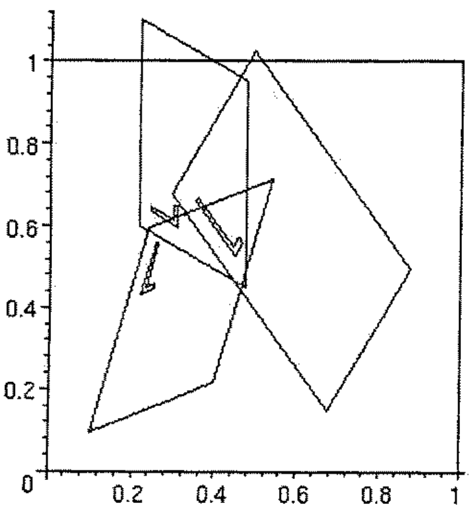
3.16

fractal template



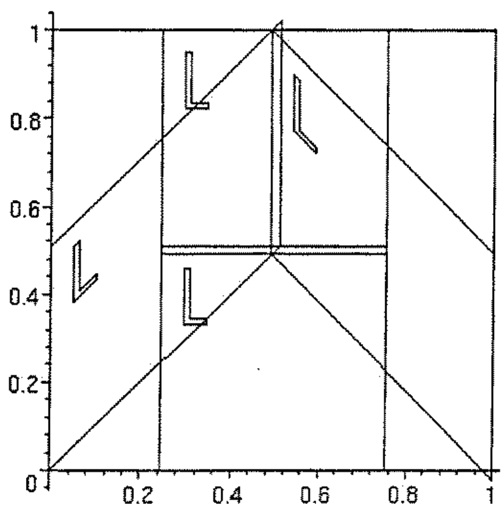
flame wave

fractal template



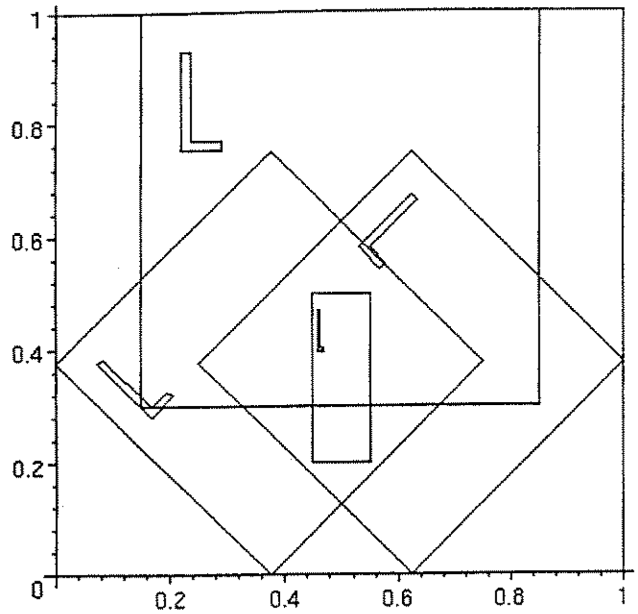
Leaves

fractal template

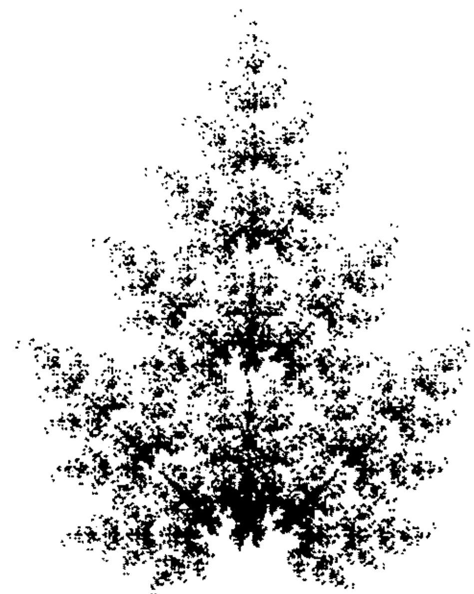


Travis' Mountain

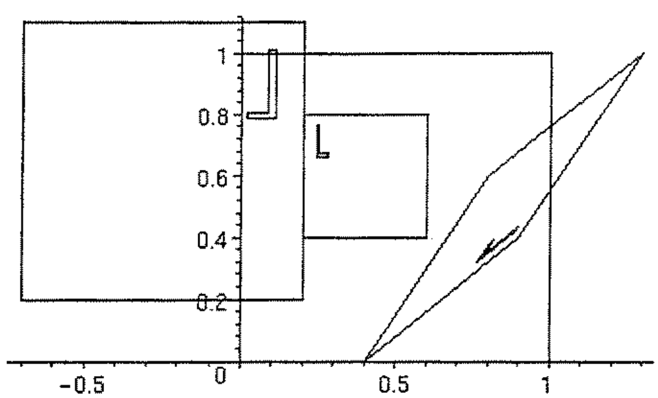
fractal template



pine tree

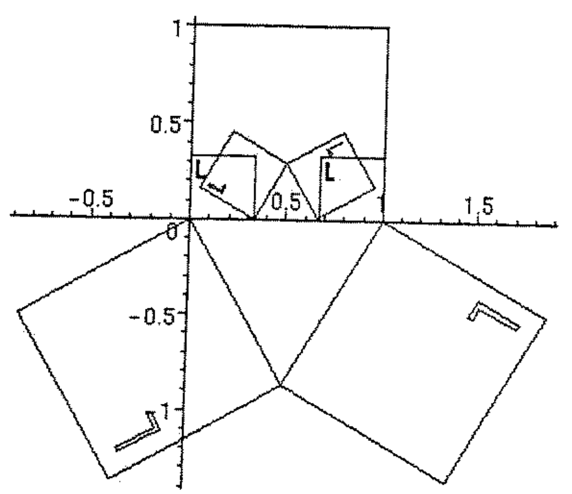


fractal template



Punk Kid

fractal template



snowflake