Name.....SOLUTIONS.....
I.D. number.....

Math 2270-1
Final Exam
December 16, 2005

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 150 points possible, and the point values for each problem are indicated in the right-hand margin. This exam counts for 30% of your final grade. Good Luck!

1) Let

$$A := \begin{bmatrix} 1 & -2 & 1 & -3 & 2 \\ -2 & 4 & -2 & 6 & -4 \\ 2 & -1 & 5 & 0 & -2 \end{bmatrix}.$$

Then $T(x) = A\,x$ is a linear a map with domain R^5 and codomain R^3. Here is the reduced row echelon form of A:

$$rref(A) = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

1a) Find a basis for the kernel of T. (5 points)

$x_5 = t$
$x_4 = s$
$x_3 = r$
$x_2 = -r - 2s + 2t$
$x_1 = -3r - s + 2t$

general sol'n to homog eqtn:

$$\vec{x} = r\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So,

basis for ker(T) is $\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

1b) Find a basis for the image of T.

(5 points)

$$\text{image}(T) = \text{span}(\text{cols}(A))$$
$$= \text{span}\left\{ \underset{\substack{\uparrow \\ \text{col}_1(A)}}{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}, \underset{\substack{\uparrow \\ \text{col}_2(A)}}{\begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}} \right\}$$

because we see from rref(A) that all other cols of A are combinations of these.
So these two cols are a basis for image(T)

1c) Express the 5th column of A as a linear combination of your basis from (1b).

(5 points)

$$\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$$

(since the same dependency holds for the cols of rref(A) as for cols(A))

1d) Find a basis for the $\underbrace{\text{orthogonal complement to the kernel of } T.}_{\text{is the rowspace}(A)!}$

(5 points)

basis is given by non-zero rows of rref(A), since row ops don't change the span of the rows.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}$$

2) Consider the matrix A below, which is invertible:

$$A := \begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{bmatrix}.$$

2a) Use the elementary row operation algorithm to find the inverse matrix, A^{-1}.

(10 points)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 3 & -5 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R_3 \\ R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 3 & -5 & -1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -3R_2+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -3 \end{array}\right]$$

$$\begin{array}{r} -4R_3+R_1 \\ 2R_3+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 5 & -4 & 12 \\ 0 & 1 & 0 & -2 & 2 & -5 \\ 0 & 0 & 1 & -1 & 1 & -3 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 7 \\ 0 & 1 & 0 & -2 & 2 & -5 \\ 0 & 0 & 1 & -1 & 1 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -2 & 7 \\ -2 & 2 & -5 \\ -1 & 1 & -3 \end{bmatrix}$$

check $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 & 7 \\ -2 & 2 & -5 \\ -1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2b) Use the adjoint formula method to recompute A^{-1}. Show enough steps so that I can verify your work.

(10 points)

$$\mathrm{cof}(A) = \begin{bmatrix} \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 4 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -2 & -1 \\ -7 & 5 & 3 \end{bmatrix}$$

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T = \begin{bmatrix} -3 & 2 & -7 \\ 2 & -2 & 5 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\det(A) = -4+4+1-2 = -1$$

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A) = -\begin{bmatrix} -3 & 2 & -7 \\ 2 & -2 & 5 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 7 \\ -2 & 2 & -5 \\ -1 & 1 & -3 \end{bmatrix}$$

3) Let W be the subspace of continuous functions with domain R and image in R, spanned by the functions $\sin(t)$, $\cos(t)$. Thus a basis for W is

$$\beta := \{\sin(t), \cos(t)\}$$

Let T: W->W be defined by

$$T(f) = (D^{(2)})(f) + 2\,D(f) + 3\,f.$$

where D stands for t-derivative. In English, that means that $T(f)$ is the sum of the second derivative of $f(t)$ with twice the first derivative of $f(t)$ with three times the function $f(t)$.

3a) Show that T is linear.

(5 points)

(i) $T(f+g) = D^{(2)}(f+g) + 2\,D(f+g) + 3(f+g)$

$= f'' + g'' + 2(f' + g') + 3f + 3g$

$= (f'' + 2f' + 3f) + (g'' + 2g' + 3g) = T(f) + T(g)$

(ii) if $k \in \mathbb{R}$, $T(kf) = (kf)'' + 2(kf)' + 3(kf)$

$= k(f'' + 2f' + 3f) = k\,T(f).$ ∎

3b) Find the matrix B for T: W->W with respect to the basis β. Recall that the derivative of $\sin(t)$ is $\cos(t)$ and the derivative of $\cos(t)$ is $-\sin(t)$.

(5 points)

If $\mathcal{B} = \{f_1, f_2\}$

then $B = [T]_{\mathcal{B}} = \Big[\, [Tf_1]_{\mathcal{B}} \;\Big|\; [Tf_2]_{\mathcal{B}} \,\Big]$

$T(\sin t) = (\sin t)'' + 2(\sin t)' + 3\sin t$

$= -\sin t + 2\cos t + 3\sin t$

$= 2\sin t + 2\cos t$

$[\ ]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ col$_1$

$T(\cos t) = -\cos t - 2\sin t + 3\cos t$

$= -2\sin t + 2\cos t$

$[\ ]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ col$_2$

So $B = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

3c) Suppose an element g of W satsifies

$$T(g(t)) = 3\ \sin(t) + 2\ \cos(t).$$

What matrix equation does the β-coordinate vector of g satisfy? Hint: it involves the matrix you found in 3b!

(5 points)

take $[\ ]_\beta$ of both sides $\Rightarrow$

$$B[g]_\beta = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3d) Solve the matrix equation in (3c), and deduce a solution g to the equation in (3c).

(5 points)

$$\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ -1/4 \end{bmatrix}$$

so $g(t) = \frac{5}{4} \sin t - \frac{1}{4} \cos t$

4a) Find an orthonormal basis for the plane of points in R^3 whose standard coordinates satisfy the equation

$$x+2y+z=0.$$

Hint: first find a basis (either by inspection or by backsolving this homogeneous equation), then use Gram-Schmidt to get an orthonormal basis.

(10 points)

$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ by inspection

$\vec{w}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\vec{z}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{w}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$\{\vec{w}_1, \vec{w}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

4b) Find the projection of the vector

$$b := \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

onto the plane of part (4a).

(5 points)

$$\begin{aligned} \text{proj}_W b &= (b \cdot \vec{w}_1)\vec{w}_1 + (b \cdot \vec{w}_2)\vec{w}_2 \\ &= \frac{1}{2}(-6) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{0}{3} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \end{aligned}$$

4c) Express the vector b above as the sum of a vector in the plane of part (4a), with a vector perpendicular to the plane. (Hint: you've done most of the work already, in part (4b)!)

(5 points)

$$\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}}_{= \text{proj}_W b} + \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{= \vec{b} - \text{proj}_W b}$$

(notice its a multiple of the plane normal vector)

5) This is a discrete dynamical system story problem: There is a famous sequence of numbers invented by and named after the historical mathematician Fibonacci. He used this sequence to model the proliferation of rabbit-like populations. The sequence begins with $r_0 = 0,\ r_1 = 1$. Then one constructs later terms in the sequence recursively, by adding the two preceding terms. In other words,

$$r_{t+1} = r_{t-1} + r_t.$$

As you can verify, this leads to the sequence

$$0,1,1,2,3,5,8,13,21,34,\ldots$$

5a) What is the next number in the Fibonacci sequence, after 34?

(2 points)

$21 + 34 = 55$

If we keep track of two Fibonacci numbers at a time, by defining

$$x(t) = \begin{bmatrix} r_t \\ r_{t+1} \end{bmatrix}$$

then

$$x(t+1) = \begin{bmatrix} r_{t+1} \\ r_{t+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} r_t \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(t),$$

with

$$x(0) = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So we have a discrete dynamical system, with the transition matrix

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Maple says:

$$eigenvectors(A) = \left(\left[\frac{1}{2} + \frac{\sqrt{5}}{2}, 1, \left\{\left[1, \frac{1}{2} + \frac{\sqrt{5}}{2}\right]\right\}\right], \left[\frac{1}{2} - \frac{\sqrt{5}}{2}, 1, \left\{\left[1, \frac{1}{2} - \frac{\sqrt{5}}{2}\right]\right\}\right]\right)$$

5b) Write the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of the two eigenvectors above.

(5 points)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\downarrow$$

$$\det = \frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} = -\sqrt{5}$$

So $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \quad ; \quad \boxed{\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}}$$

Repeating from the previous page, for

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Maple says:

$$eigenvectors(A) = \left(\left[\frac{1}{2} + \frac{\sqrt{5}}{2}, 1, \left\{ \left[1, \frac{1}{2} + \frac{\sqrt{5}}{2} \right] \right\} \right], \left[\frac{1}{2} - \frac{\sqrt{5}}{2}, 1, \left\{ \left[1, \frac{1}{2} - \frac{\sqrt{5}}{2} \right] \right\} \right] \right)$$

5c) Use the Maple output above, your work in (5b), and your knowledge of discrete dynamical systems to find a closed form expression for x(t) and for its first component, the Fibonacci number r_t.

(8 points)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

so $$A^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^t}_{\lambda_1^t} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^t}_{\lambda_2^t} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

in particular, get r_t from 1^{st} comp of $A^t \vec{x}_0$:

$$r_t = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^t - \left(\frac{1-\sqrt{5}}{2}\right)^t \right]$$

check!

$$r_0 = 0$$
$$r_1 = \frac{1}{\sqrt{5}}(\sqrt{5}) = 1$$
$$r_2 = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2 \right)$$
$$= \frac{1}{\sqrt{5}} \left(\underbrace{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)}_{\sqrt{5}} \right) \left(\underbrace{\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}}_{1} \right) = 1 \checkmark$$

note, since the second term $\frac{1}{\sqrt{5}}(-1)\left(\frac{1-\sqrt{5}}{2}\right)^t$ is between $-.5$ and $.5$ $\forall t \geq 0$ (and $\to 0$ rapidly as $t \to \infty$), you can obtain r_t by rounding of the 1^{st} term $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^t$ to the nearest integer

e.g. $r_{10} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{10} \cong 55.0036\ldots$, or $r_{20} = 6765$, etc.

so $r_{10} = 55$, see page previous!

6) Find an eigenbasis (of R^2) for the matrix

$$A := \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

(15 points)

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda-3)^2 - 2^2 = (\lambda-3-2)(\lambda-3+2)$$
$$= (\lambda-5)(\lambda-1)$$

evals $\lambda = 1, 5$

E_1:

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

since A is symmetric, $E_5 \perp E_1$

so E_5 basis is $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$ will do

7a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

$$* \qquad x^T A\,x + B\,x + c = 0.$$

into one without any ''cross terms''. Recall that in the equation above, x represents the standard basis coordinate vector of a point in n-space (written as a column vector), A is a symmetric n by n matrix, B is a row vector (1 by n matrix) and c is a scalar. Be precise in explaining how one finds a new orthonormal basis for R^n so that the standard coordinates of points satisfy the original equation if and only if the coordinates with respect to the new basis satisfy an equation with no cross terms. What theorem about symmetric matrices makes this possible? Show how the new equation is derived from the original one.

(4 points)

A is symmetric so spectral theorem says

$\exists$ o.n. eigenbasis $\mathcal{B} = \{\vec{w}_1, \ldots \vec{w}_n\}$ and so orthog matrix $S = S_{E \leftarrow \mathcal{B}} = \left[\vec{w}_1 \middle| \cdots \vec{w}_n\right]$

so $\vec{x} = S\vec{u}$ where $\vec{u} = [\vec{x}]_{\mathcal{B}}$.

substituting into $*$

$$u^T \underbrace{S^T A S}_{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} u + BSu + c = 0$$

gives eqtn w/o cross terms, for the new coords

7b) Apply the procedure from part (7a) to find a rotated basis of R^2 so that the conic section whose standard coordinates satisfy

$$3\,x^2 + 4\,x\,y + 3\,y^2 = 20$$

has coordinates $\begin{bmatrix} u \\ v \end{bmatrix}$ with respect to the new basis so that the equation these new coordinates satisfy has no $u\,v$ cross term. Put the resulting equation into standard form and identify the conic. (HINT: Refer to your work in problem 6 !!!)

(8 points)

$$[x, y]\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 20$$

from #6, $\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = S\begin{bmatrix} u \\ v \end{bmatrix}$$

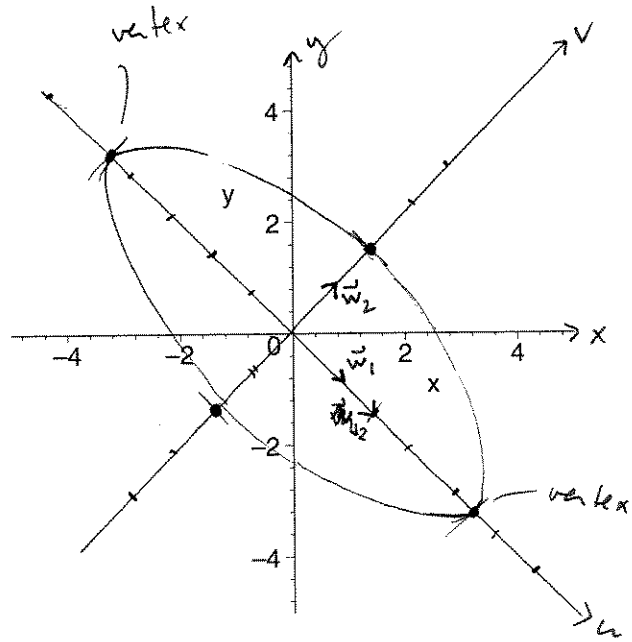
$$[u \ v]\underbrace{S^T A S}_{\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}}\begin{bmatrix} u \\ v \end{bmatrix} = 20$$

$$u^2 + 5v^2 = 20$$

$$\frac{u^2}{20} + \frac{v^2}{4} = 1 \quad \text{ellipse.}$$

7c) Carefully sketch the rotated basis vectors you found in part (7b), the new coordinate axes, and the conic section you found in (7b). Label the vertex or vertices and exhibit the $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ coordinates of one vertex.

(8 points)



$$\frac{u^2}{20} + \frac{v^2}{4} = 1$$

vertex @

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \pm 2\sqrt{5} \\ 0 \end{bmatrix} \approx \begin{bmatrix} 4.5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \pm 2\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \pm \begin{bmatrix} \sqrt{10} \\ -\sqrt{10} \end{bmatrix}$$

8) True-False. Four points each (two points for answer, two points for justification.)

(20 points)

8i) If A and B are square matrices, then

$$(A+B)^2 = A^2 + 2AB + B^2.$$

FALSE! $AB \neq BA$ in general, and

$$(A+B)^2 = (A+B)(A+B) = (A+B)A + (A+B)B$$
$$= A^2 + BA + AB + B^2$$

8ii) Let A be an invertible square matrix. Then the equation $Ax = Ay$ implies that $x = y$.

TRUE. $Ax = Ay$ and A^{-1} exists

$$\Rightarrow A^{-1}(Ax) = A^{-1}(Ay)$$
$$\Rightarrow Ix = Iy$$
$$\Rightarrow x = y$$

8iii) If B is a square matrix and its rows are orthonormal, then B is orthogonal.

TRUE ~~yes~~: if B has rows orthonormal, then $(BB^T)_{ij} = row_i(B) \cdot row_j(B) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

so $BB^T = I$

so B^{-1} exists & $= B^T$

so $B^T B = I$ & cols are orthonormal, so B is orthogonal.

8iv) If T is the projection in R^3 to the plane $x+y+z=0$, then there is a basis for R^3 for which the matrix of this projection transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

TRUE.

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

where $\vec{v}_1, \vec{v}_2$ are in the plane

& $\vec{v}_3$ is $\perp$ to the plane. Then $\left.\begin{array}{l} T\vec{v}_1 = \vec{v}_1 \\ T\vec{v}_2 = \vec{v}_2 \\ T\vec{v}_3 = \vec{0} \end{array}\right\} \Rightarrow [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8v) The addition angle formula for $\cos(\alpha+\beta)$ is:

$$\cos(\alpha+\beta) = \cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta).$$

False! $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

since is 1,1 entry of $\underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ +\sin\alpha & \cos\alpha \end{bmatrix}}_{[Rot_\alpha]} \underbrace{\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}}_{[Rot_\beta]}$!