Name ....SOLUTIONS....

I.D. number....

Math 2270-1
Exam 2
November 4, 2005

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. Good Luck!

1a) Consider the matrix system

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

Find the least squares solution to this problem.

(10 points)

Normal eqtn:

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{33-9}\begin{bmatrix} 3 & -3 \\ -3 & 11 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix} 0 \\ -18+66 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

1b) The matrix system in part (1a) could have arisen as a least squares problem in the context of trying to get a best-line approximation to a data set consisting of the 3 points

$$\left\{\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}\right\}.$$

Explain the meaning of "r" and "s" in this case. Then, sketch the three points below and the least-squares line fit.

(10 points)

want $y = mx + b$

$$\begin{bmatrix}3\\0\\3\end{bmatrix} = m\begin{bmatrix}-1\\1\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix}$$

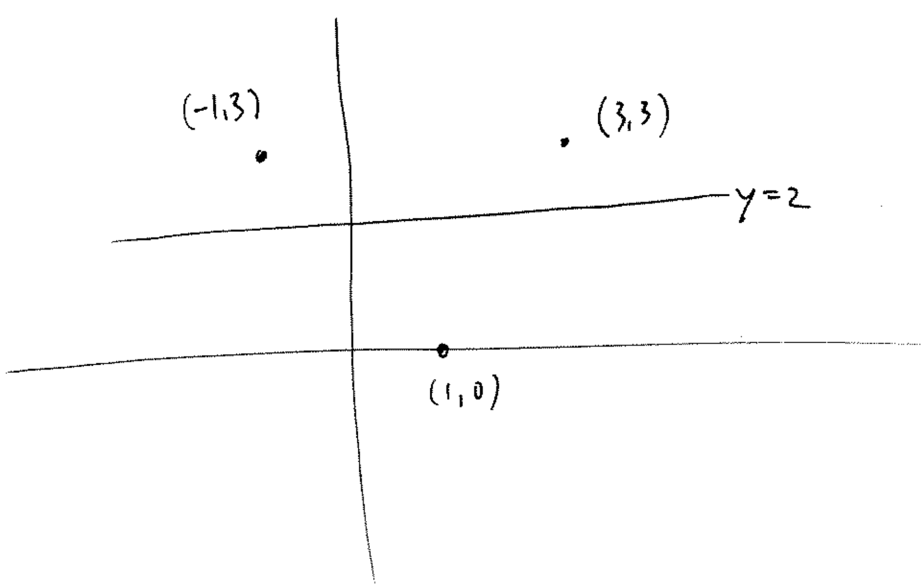
or, reordering

$$m\begin{bmatrix}1\\-1\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\3\\3\end{bmatrix}$$

so $r = m = \text{slope}$
$s = b = \text{intercept}$

since sol'n is $\begin{bmatrix}r\\s\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$

deduce line is $y = 0x + 2 = 2$



2a) Define the *image of T*, and the *kernel of T*, for T: V--> W a linear transformation. (6 points)

$\text{image}(T) := \{ w \in W \text{ s.t. } \exists v \in V \text{ with } T(v) = w \}$

$\text{kernel}(T) := \{ v \in V \text{ s.t. } T(v) = 0 \}$

2b) For a linear transformation T:V->W, prove that the image of T is a subspace. (6 points)

(i) image(T) closed under addition:

Let $w, z \in \text{image}(T)$

Thus $\exists\, u \in V,\ T(u) = w$

$v \in V,\ T(v) = z$

Then $T(u+v) = T(u) + T(v)$ (T linear)

$= w + z$

so $w + z \in \text{image}(T)$

(ii) image(T) closed under scalar mult:

Let $w \in \text{image}(T)$, $k \in \mathbb{R}$.

Then $\exists\, u \in V,\ T(u) = w$. Then $T(ku) \overset{T \text{ linear}}{=} kT(u) = kw$, so $kw \in \text{image}(T)$

2c) Let $\{f_1, f_2, f_3\}$ be a set of three linearly independent vectors in a linear space V. Suppose that the element g of V is not in the span of $\{f_1, f_2, f_3\}$. Prove that $\{g, f_1, f_2, f_3\}$ is a set of four linearly independent vectors. (8 points)

Let $c_0 g + c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

Case I: if $c_0 = 0$ then $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

so $c_1 = c_2 = c_3 = 0$ since f_1, f_2, f_3 lin ind.

so $c_0 = c_1 = c_2 = c_3 = 0$

Case II: if $c_0 \neq 0$, then

$c_0 g = -c_1 f_1 - c_2 f_2 - c_3 f_3$

$g = -\frac{c_1}{c_0} f_1 - \frac{c_2}{c_0} f_2 - \frac{c_3}{c_0} f_3$

But this case cannot occur because we assumed g was <u>not</u> in the span of f_1, f_2, f_3.

Thus only case I occurs, and $c_0 = c_1 = c_2 = c_3 = 0$ so $\{g, f_1, f_2, f_3\}$ is a linearly independent set

3) Let V be the two-dimensional function vector space with basis $\beta = \{\mathbf{e}^t, \mathbf{e}^{(-t)}\}$. Let T be the linear transformation from V to V defined by

$$T(f) = f' + 3f$$

("T of f is the derivative of f plus 3 times f").

3a) Find the matrix B for T with respect to the basis β. (Hint: it's diagonal!) (6 points)

$$T(e^t) = 4e^t + 0e^{-t} \quad ; \quad T(e^{-t}) = -e^{-t} + 3e^{-t} = 0e^t + 2e^{-t}$$

$$\text{so} \quad B = \left[[T(e^t)]_\beta \,\middle|\, [T(e^{-t})]_\beta \right] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

3b) Another basis for V is given by $\kappa = \{\cosh(t), \sinh(t)\}$. Recall that $\cosh(t) = \frac{1}{2}\mathbf{e}^t + \frac{1}{2}\mathbf{e}^{(-t)}$ and $\sinh(t) = \frac{1}{2}\mathbf{e}^t - \frac{1}{2}\mathbf{e}^{(-t)}$. What is the change of basis matrix S which converts κ-coordinates into β-coordinates? (7 points)

$$S = \left[[\cosh t]_\beta \,\middle|\, [\sinh t]_\beta \right] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

3c) Find the matrix for the linear tranformation T(f), with respect to the basis κ. (7 points)

$$[T]_\kappa = S^{-1} B \, S_{\beta \leftarrow \kappa}$$

$$= -2 \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

check: $T(\cosh t) = \sinh t + 3\cosh t$
$T(\sinh t) = \cosh t + 3\sinh t$ ✓

4a) Find an orthonormal basis for R^3 by using the Gram-Schmidt algorithm on the three vectors

$$\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Hint: The first two vectors in the set are already orthogonal to each other.

(10 points)

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{since } \vec{v}_2 \perp \vec{w}_1)$$

$$\left\{\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\right\}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3\cdot\vec{w}_1)\vec{w}_1 - (\vec{v}_3\cdot\vec{w}_2)\vec{w}_2 \,/\, \|\vec{?}_0\|$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}(-1)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,/\, \|\vec{?}_0\| = \frac{1}{\|\vec{?}_0\|}\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

4b) What is the $A = QR$ factorization for the matrix

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}?$$

(10 points)

$$\vec{v}_1 = \sqrt{2}\,\vec{w}_1$$
$$\vec{v}_2 = \vec{w}_2$$
$$\vec{v}_3 = (\vec{v}_3\cdot\vec{w}_1)\vec{w}_1 + (\vec{v}_3\cdot\vec{w}_2)\vec{w}_2 + (\vec{v}_3\cdot\vec{w}_3)\vec{w}_3$$
$$= -\frac{1}{\sqrt{2}}\vec{w}_1 + 1\cdot\vec{w}_2 + \frac{1}{\sqrt{2}}\vec{w}_3$$

So

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{2} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

$$\begin{array}{ccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{array}$$

(alternate way to get R:

$A = QR$

$\Rightarrow Q^TA = Q^TQR = IR = R.$;

$$Q^TA = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

5) True-False: 5 points for each problem; two points for the correct answer and three points for the explanation.

(20 points)

5a) If {u,v,w,z} is an orthonormal collection of vectors, then ||u+v+w+z||=2.

TRUE! $\|u+v+w+z\|^2 = (\vec{u}+\vec{v}+\vec{w}+\vec{z})\cdot(\vec{u}+\vec{v}+\vec{w}+\vec{z})$

$= \vec{u}\cdot(\vec{u}+\vec{v}+\vec{w}+\vec{z}) + \vec{v}\cdot(\text{''}) + \vec{w}\cdot(\text{''}) + \vec{z}\cdot(\text{''})$

$= \vec{u}\cdot\vec{u} + 0 + \vec{v}\cdot\vec{v} + 0 + \vec{w}\cdot\vec{w} + 0 + \vec{z}\cdot\vec{z} + 0$

$= 1+1+1+1 = 4.$

So $\|\vec{u}+\vec{v}+\vec{w}+\vec{z}\| = \sqrt{4} = 2.$

5b) If A is an invertible matrix, then A^T is too. In fact,

$$[A^T]^{-1} = [A^{-1}]^T.$$

TRUE! $AA^{-1} = I$, $A^{-1}A = I$

$\Rightarrow \underbrace{(AA^{-1})^T}_{(A^{-1})^T A^T} = I^T = I$ and $\underbrace{(A^{-1}A)^T}_{A^T(A^{-1})^T} = I$ so $(A^T)^{-1} = (A^{-1})^T$!

5c) If T is the projection in R^3 to the plane $x+y+z=0$, then there is a basis for R^3 for which the matrix of this projection transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

TRUE! pick $\vec{v}_1, \vec{v}_2$ in the plane (and ind.), so $T\vec{v}_1 = \vec{v}_1$, $T\vec{v}_2 = \vec{v}_2$

pick $\vec{z} \perp$ to the plane, so $T\vec{z} = \vec{0}$

then $[T]_{\{\vec{v}_1, \vec{v}_2, \vec{z}\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

5d) If the dimension of V is 6 and the dimension of W is 2, and if L:V-->W is linear, then the dimension of the kernel of L is at most 4.

$\dim(\ker L) + \underbrace{\dim(\operatorname{Im} L)}_{\leq 2} = \dim V = 6$

$\Rightarrow \dim(\ker L) \geq 4$, not ≤ 4; so FALSE!

e.g. $L(v) = 0 \ \forall \vec{v}$, all of V is in the kernel,

so $\dim(\ker L) = 6.$