Name...
I.D. number...

Math 2270-1
Exam 2
November 4, 2005

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. Good Luck!

1a) Consider the matrix system

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

Find the least squares solution to this problem.

(10 points)

1b) The matrix system in part (1a) could have arisen as a least squares problem in the context of trying to get a best-line approximation to a data set consisting of the 3 points

$$\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right\}.$$

Explain the meaning of "r" and "s" in this case. Then, sketch the three points below and the least-squares line fit.

(10 points)

2a) Define the *image of T,* and the *kernel of T,* for T: V--> W a linear transformation.

(6 points)

2b) For a linear transformation T:V->W, prove that the image of T is a subspace.

(6 points)

2c) Let $\{f_1, f_2, f_3\}$ be a set of three linearly independent vectors in a linear space V. Suppose that the element g of V is not in the span of $\{f_1, f_2, f_3\}$. Prove that $\{g, f_1, f_2, f_3\}$ is a set of four linearly independent vectors.

(8 points)

3) Let V be the two-dimensional function vector space with basis $\beta = \{\mathbf{e}^t, \mathbf{e}^{(-t)}\}$. Let T be the linear transformation from V to V defined by

$$T(f) = f' + 3f$$

("T of f is the derivative of f plus 3 times f").

3a) Find the matrix B for T with respect to the basis β. (Hint: it's diagonal!)

(6 points)

3b) Another basis for V is given by $\kappa = \{\cosh(t), \sinh(t)\}$. Recall that $\cosh(t) = \frac{1}{2}\mathbf{e}^t + \frac{1}{2}\mathbf{e}^{(-t)}$ and $\sinh(t) = \frac{1}{2}\mathbf{e}^t - \frac{1}{2}\mathbf{e}^{(-t)}$. What is the change of basis matrix S which converts κ-coordinates into β-coordinates?

(7 points)

3c) Find the matrix for the linear tranformation T(f), with respect to the basis κ.

(7 points)

4a) Find an orthonormal basis for R^3 by using the Gram-Schmidt algorithm on the three vectors

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Hint: The first two vectors in the set are already orthogonal to each other.

(10 points)

4b) What is the $A = Q\,R$ factorization for the matrix

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}?$$

(10 points)

5) True-False: 5 points for each problem; two points for the correct answer and three points for the explanation.

(20 points)

5a) If {u,v,w,z} is an orthonormal collection of vectors, then ||u+v+w+z||=2.

5b) If A is an invertible matrix, then A^T is too. In fact,

$$[A^T]^{-1} = [A^{-1}]^T.$$

5c) If T is the projection in R^3 to the plane $x+y+z=0$, then there is a basis for R^3 for which the matrix of this projection transformation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

5d) If the dimension of V is 6 and the dimension of W is 2, and if L:V-->W is linear, then the dimension of the kernel of L is at most 4.