Name........SOLUTIONS........

I.D. number..

Math 2270-1
First Exam
September 27, 2005

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

1) Consider the following matrix equation,

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

1a) Find the solution to this system by writing the appropriate augmented matrix and computing its reduced row echelon form. Use the same algorithm we use for much larger systems and show all steps, because part of your credit on this problem arises from showing that you understand the general "rref" procedure.

(8 points)

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ -2 & 1 & -3 \end{array}\right]$$

$$\begin{array}{l} \\ 2R_1+R_2 \end{array}\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/5 \end{array}\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} -2R_2+R_1 \\ \end{array}\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

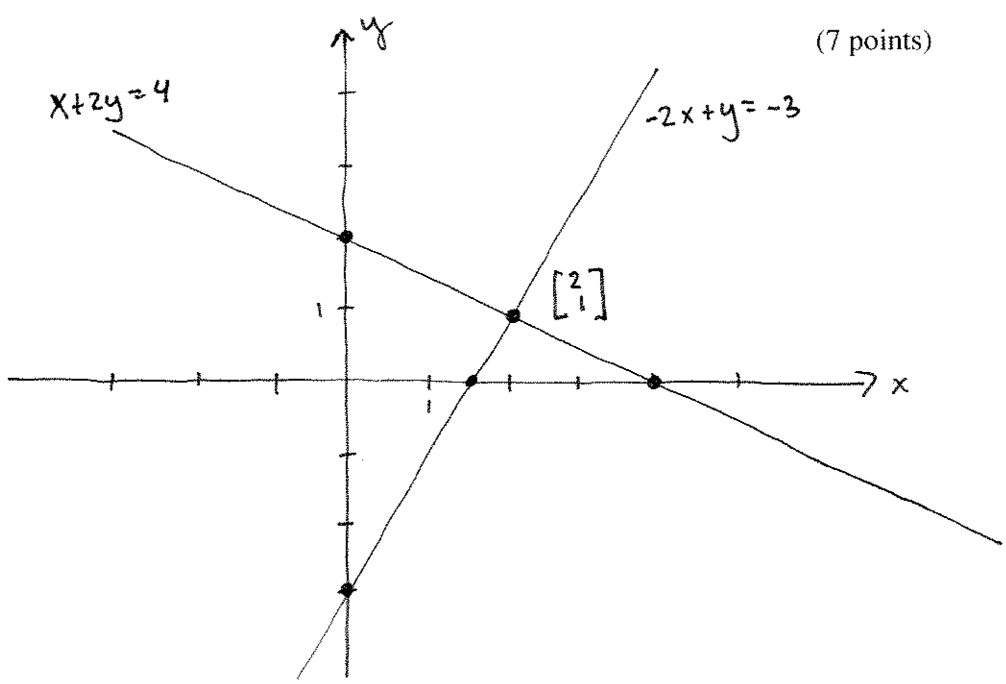
Sol'n $x=2$, $y=1$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

1b) One geometric interpretation of the matrix system above is that you are finding the intersection of two lines in the plane. Sketch and label the two lines and their intersection point, which should be the solution you found in (1a):

(7 points)

$x+2y=4$
$-2x+y=-3$

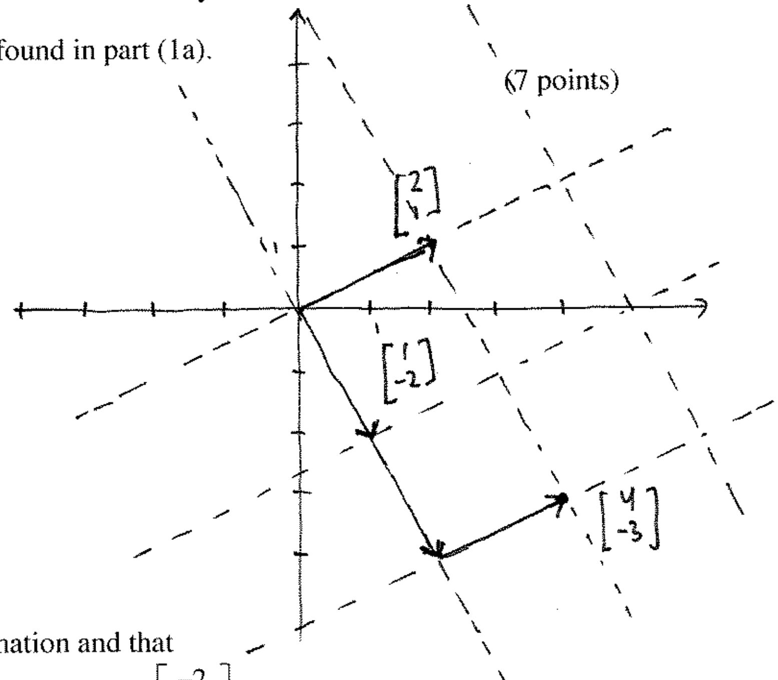


1c) The other geometric interpretation of the matrix system in this problem is asking how to express the vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ as a linear combination of two other vectors. Write the system in linear combination form and draw a picture which illustrates the solution you found in part (1a).

(7 points)

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(system was
$$x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$
)



1d) Suppose now that f: $R^2 \to R^3$ is a linear transformation and that

$$f\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \text{ and } f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}.$$

Use your work from part (1c) to deduce the value of $f\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$

(8 points)

$$f\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right) = f\left(2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

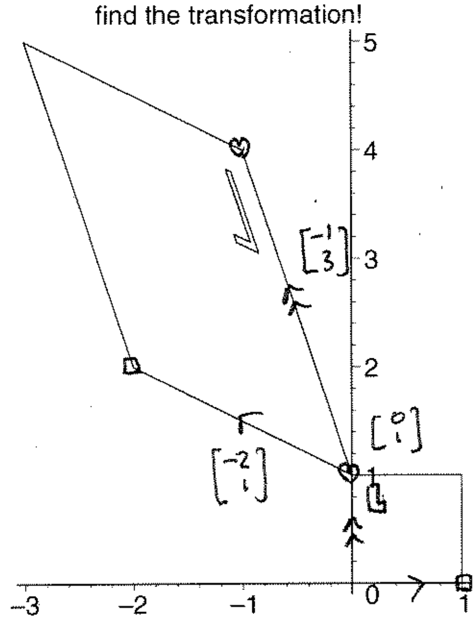
$$= 2f\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) + f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$ because f is linear.

$$= 2\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$$

2) Here is an affine transformation picture, which depicts the unit square and its image after applying a certain affine map. Find the transformation!

(10 points)



$$f\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$
$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-2\\2\end{bmatrix}$$
$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}$$

$$f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=x\begin{bmatrix}a\\b\end{bmatrix}+y\begin{bmatrix}c\\d\end{bmatrix}+\begin{bmatrix}e\\f\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}e\\f\end{bmatrix}$$
$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$$
$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}c\\d\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}$$

So, $\boxed{f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}-2&-1\\1&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}}$

3) Here is a matrix and its reduced row echelon form:

$$A := \begin{bmatrix} 1 & 2 & 0 & 3 & 3 \\ -1 & 1 & -3 & 2 & -1 \\ 3 & 3 & 3 & 3 & 6 \\ 2 & 1 & 3 & -2 & 1 \end{bmatrix} \qquad RREF := \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3a) Find a basis for the image of A. Justify your work. (8 points)

$col_3(A) = 2\,col_1(A) - col_2(A)$ so $col_3(A)$ is redundant

$col_5(A) = 2\,col_1(A) - col_2(A) + col_4(A)$ so $col_5(A)$ "

cols 1, 2, 4 of A are independent, and span the image of A

all dependencies & independencies for cols can be read from rref(A), because column dependencies correspond to sol'ns of homogeneous systems, which are unchanged under elementary row ops.

$$\boxed{\text{so basis for image}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \\ -2 \end{bmatrix} \right\}}$$

3b) Find a basis for the kernel of A. Justify your work. (8 points)

backsolve: $\text{rref}\,A \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right.$

$x_5 = s$

$x_4 = -s$

$x_3 = t$

$x_2 = t + s$

$x_1 = -2t - 2s$

$$\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\boxed{\text{So basis for kernel}(A) \text{ is } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

span

independent: if $s \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

then entry 5 $\Rightarrow s = 0$

entry 3 $\Rightarrow t = 0$

$$A := \begin{bmatrix} 1 & 2 & 0 & 3 & 3 \\ -1 & 1 & -3 & 2 & -1 \\ 3 & 3 & 3 & 3 & 6 \\ 2 & 1 & 3 & -2 & 1 \end{bmatrix} \qquad RREF := \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3c) Express the fifth column of A as a linear combination of your basis image basis from (3a).

(7 points)

already done. (& explained in 3a)

$$col_5(A) = 2col_1(A) - col_2(A) + col_4(A)$$

$$\begin{bmatrix} 3 \\ -1 \\ 6 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 3 \\ 8 \end{bmatrix}$$

3d) State the rank + nullity theorem for matrix transformations. Verify that the theorem holds in this example.

(7 points)

rank + nullity = dim of domain
= # of cols in matrix

$$\underbrace{\text{rank}}_{\dim(\text{Image } f)} + \underbrace{\text{nullity}}_{\dim(\text{kernel } f)}$$

3 + 2 = 5.

4) True-False. Two points for correct answer, and three points for justification, on each problem.
(30 points)

4a) If A,B,C,D are matrices, and if their dimensions are appropriate to the indicated multiplications, then it is always true that

$$(A+B)(C+D)=AC+BC+AD+BD.$$

TRUE

$$= (A+B)C + (A+B)D$$
$$= AC+BC+AD+BD$$

distrib prop of mult over +

4b) If four vectors span R^4 then they are a basis for R^4.

true: $\mathbb{R}^4$ is 4-dim, so spanning set of 4 vectors is independent

$\Longrightarrow$ basis

OR, $\text{rref}\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3 \middle| \vec{v}_4\right] = I$ since they span.

this also implies independence

4c) If A is a two by two square matrix and if

$$A^2 = A$$

then A is either the identity matrix or the zero matrix.

FALSE : e.g. projection onto x-axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4d) If the vectors **u**, **v**, **w** are linearly dependent then **w** is a linear combination of **u** and **v**.

FALSE (the dependency may not involve w)

e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\quad\;\;\vec{u} \qquad \vec{v} \qquad \vec{w}$

$-2\vec{u} + \vec{v} = \vec{0}$

4e) If W is a subspace of dimension k then any collection of more than k vectors from W spans W.

False.

e.g. $W = \mathbb{R}^2$

$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix} \right\}$

4f) If A is an invertible n by n matrix, then the images of A and its inverse matrix coincide.

True A is invertible means $\text{image}(A) = \mathbb{R}^n$

same is true for A^{-1}, since $(A^{-1})^{-1} = A$.