Math 2270-1
Tuesday 6 Dec.

§8.3 Singular values and Singular value decomposition of a matrix transformation

(Singular ↔ stretching)

This actually relates to 1st MAPLE project, where you constructed fractals by iterating a set mapping, using a finite # of contractions, as we shall see.

Let's start generally...

Consider $T: \mathbb{R}^n \to \mathbb{R}^m$
$$T(\vec{x}) = A_{m\times n}\vec{x}$$

for $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0}$, define $\dfrac{\|T(\vec{v})\|}{\|\vec{v}\|} := \sigma(\vec{v})$; the stretching factor of T (or A) in the $\vec{v}$ direction
(notice this factor does only depend on the direction, not the magnitude, of $\vec{v}$)

$\sigma(\vec{v}) \geq 0$;
$\sigma(\vec{v}) = 0$ iff $\vec{v} \in \ker(A)$. $\quad \dim(\ker(A)) = n - r$
$r = \text{rank}(A)$.

If we replace $\vec{v}$ with $\dfrac{\vec{v}}{\|\vec{v}\|}$, a unit vect in same dir,

then
$$\sigma(\vec{v}) = \|A\vec{v}\|$$
$$\sigma^2(\vec{v}) = A\vec{v}\cdot A\vec{v}$$
$$= (A\vec{v})^T(A\vec{v})$$
$$= \vec{v}^T\underbrace{(A^TA)}\vec{v}$$

symmetric! Apply spectral Theorem
$n\times n$

$\exists\ S = [\vec{w}_1|\vec{w}_2|\cdots\vec{w}_n]$ orthog matrix of evects of A^TA, with evals $\lambda_1, \lambda_2 \ldots \lambda_n$

So $\sigma^2(\vec{w}_j) = \vec{w}_j^T(A^TA\vec{w}_j) = \vec{w}_j\cdot\lambda_j\vec{w}_j$
$= \lambda_j$

$\boxed{\text{We define } \sigma_j = \sqrt{\lambda_j} = \text{stretch factor of } A \text{ in } \vec{w}_j \text{ direction} = \text{singular value}}$

notice, every $v \in \ker(A)$ is a $\lambda = 0$ evect (iff)

For any $\vec{v} = S\vec{u}$ with $\|\vec{v}\| = \|\vec{u}\| = 1$,
$$\sigma^2(\vec{v}) = \vec{u}^T\underbrace{S^TA^TAS}_{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}}\vec{u} = \lambda_1u_1^2 + \lambda_2u_2^2 + \cdots + \lambda_nu_n^2.$$

If we order eigen basis so $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

then $\lambda_n \leq \sigma^2(\vec{v}) \leq \lambda_1$
$\sigma_n \leq \sigma(\vec{v}) \leq \sigma_1$

(only $\lambda_1 \cdots \lambda_r \neq 0$)

Example

$$A = \begin{bmatrix} -1/4 & -3/4 \\ -1/4 & 3/4 \end{bmatrix}$$

Find the singular values of A:

$$A^T A = \begin{bmatrix} -1/4 & -1/4 \\ -3/4 & 3/4 \end{bmatrix}\begin{bmatrix} -1/4 & -3/4 \\ -1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ 0 & 9/8 \end{bmatrix}$$

$$\sigma_1 = \sqrt{9/8} \qquad \lambda_1 = 9/8 \qquad E_{9/8} = \text{span}\left\{\overset{\vec{w}_1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\right\}$$

$$\sigma_2 = \sqrt{1/8} \qquad \lambda_2 = 1/8 \qquad E_{1/8} = \text{span}\left\{\underset{\vec{w}_2}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\right\}.$$

$\sigma_1 > 1$.
So $f(\vec{x}) = A\vec{x} + \vec{b}$
is not a contraction, see next page!

There's more to see.

$$\left.\begin{array}{l} A\vec{w}_1 = A\vec{e}_2 = \begin{bmatrix} -3/4 \\ 3/4 \end{bmatrix} \\ A\vec{w}_2 = A\vec{e}_1 = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix} \end{array}\right\} \quad A\vec{w}_j \perp A\vec{w}_k \quad (k \neq j) \text{ always,}$$

since $A\vec{w}_j \cdot A\vec{w}_k = w_j^T \underbrace{A^T A\, \vec{w}_k}_{\lambda_k \vec{w}_k}$

In general case,

$$f(\vec{x}) = A_{m\times n}\vec{x}$$

deduce $A\vec{w}_1, \cdots A\vec{w}_r$ are $\neq 0$, orthog in codomain $(A\vec{w}_{r+1}, \cdots A\vec{w}_n = 0)$.

Define $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{w}_1$, $\vec{u}_2 = \frac{1}{\sigma_2}A\vec{w}_2, \ldots \vec{u}_r = \frac{1}{\sigma_r}A\vec{w}_r$, complete to o.n. basis $\vec{u}_{r+1}, \cdots \vec{u}_m$ of codomain.

$$A\,\overbrace{\left[\vec{w}_1 \middle| \vec{w}_2 \middle| \cdots \vec{w}_n\right]}^{S}_{n\times n} = \left[\sigma_1\vec{u}_1 \middle| \sigma_2\vec{u}_2 \middle| \cdots \sigma_r\vec{u}_r \middle| 0 \middle| 0 \middle| \cdots\right]$$

$$= \underbrace{\left[\vec{u}_1 \middle| \vec{u}_2 \middle| \cdots \middle| \vec{u}_m\right]_{m\times m}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \sigma_r & \\ & & & & 0 \; 0 \end{bmatrix}_{m\times n}}_{\Sigma}$$

or, $\boxed{A = U_{m\times m}\,\Sigma_{m\times n}\,S^T_{n\times n}}$ Singular value decomposition

$U_{m\times m}$: orthog in codomain

$\Sigma_{m\times n}$: stretch matrix from domain to codomain

$S^T_{n\times n}$: orthog in domain

Deamon

If you have loaded the procedures from the Lpictures.mws file, you can try defining some affine contractions, draw the L-picture, and then generate the fractal. We show it here for Sierpinski's triangle.

```
> f1:=P->AFFINE1(P,1/6,-.5,1/6,.5,1/3,.25);
  f2:=P->AFFINE1(P,-1/6,-.5,-1/6,.5,2/3,.25);
  f3:=P->AFFINE1(P,.25,-.75,.25,.75,0,.25)
  f4:=P->AFFINE1(P,-.25,-.75,-.25,.75,1,.25);
```

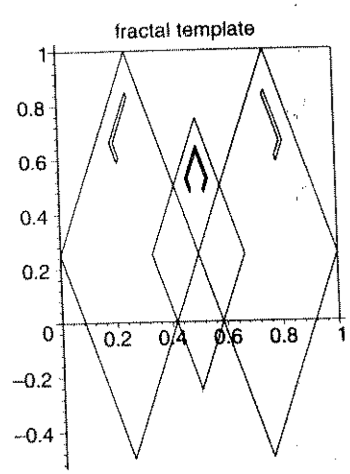
$$f1 := P \to \mathrm{AFFINE1}\left(P, \frac{1}{6}, -0.5, \frac{1}{6}, 0.5, \frac{1}{3}, 0.25\right)$$

$$f2 := P \to \mathrm{AFFINE1}\left(P, \frac{-1}{6}, -0.5, \frac{-1}{6}, 0.5, \frac{2}{3}, 0.25\right)$$

$$f3 := P \to \mathrm{AFFINE1}(P, 0.25, -0.75, 0.25, 0.75, 0, 0.25)$$

$$f4 := P \to \mathrm{AFFINE1}(P, -0.25, -0.75, -0.25, 0.75, 1, 0.25)$$

```
> TESTMAP([f1,f2,f3,f4]);
```



Since the template is correct, we may proceed.

```
> S:={[0,0]}:#initial set consisting of one point
> 4^8; #good to keep point numbers well below 100,000,
       #so as not to strain Maple's memory
```

65536

Based on the computation above I will do nine iterations below:

```
> for i from 1 to 8 do
  S1:=map(f1,S);
  S2:=map(f2,S);
  S3:=map(f3,S);
  S4:=map(f4,S);
  S:=`union`(S1,S2,S3,S4);
  od:
> pointplot(S,symbol=point,scaling=constrained,
      axes=none,title=`Jeff and Scott's Deamon`);
```

But wait, is this a fractal?? Is each map a contraction??

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotectec
> A:=matrix(2,2,[-.25,-.75,-.25,.75]);
```

$$A := \begin{bmatrix} -0.25 & -0.75 \\ -0.25 & 0.75 \end{bmatrix}$$

```
> eigenvals(transpose(A)&*A);
```

0.1250, 1.125

Oh Oh!

Find the SVD for $A = \begin{bmatrix} -1/4 & -3/4 \\ -1/4 & 3/4 \end{bmatrix}$

from page 2, $\vec{w}_1 = \vec{e}_2$, $\vec{w}_2 = \vec{e}_1$

$$S = \begin{array}{cc} \vec{w}_1 & \vec{w}_2 \\ \downarrow & \downarrow \end{array} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A\vec{w}_1 = \begin{bmatrix} -3/4 \\ 3/4 \end{bmatrix} \parallel \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A\vec{w}_2 = \begin{bmatrix} -1/4 \\ -1/4 \end{bmatrix} \parallel \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

$$U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$A = U \Sigma S^T$$

$$A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\uparrow \text{ rotate}} \underbrace{\begin{bmatrix} \sqrt{9/8} & 0 \\ 0 & \sqrt{1/8} \end{bmatrix}}_{\uparrow \text{ scale}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\uparrow \text{ reflect}}$$