Math 2270-1
Monday Dec 5 §8.2

The amazing and powerful...

Spectral Theorem : Let $A_{n\times n}$ be a symmetric, real-entry matrix.
Then $\exists$ an orthonormal eigenbasis (of $\mathbb{R}^n$) for A.
i.e. $\exists$ S orthogonal with $S^{-1}AS = S^TAS$ diagonal.

Example : $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\lambda = 0, 3$

easy to check. : $E_0 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, $E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Note: $E_0 \perp E_3$ (check!)

not accidental : $\boxed{\text{eigenspaces of symmetric matrices are always mutually orthogonal:}}$

If $A^T = A$
$A\vec{v} = \lambda_1 \vec{v}$
$A\vec{w} = \lambda_2 \vec{w}$ $\quad \lambda_1 \neq \lambda_2$

then $\vec{w}\cdot A\vec{v} = \lambda_1 \vec{w}\cdot\vec{v}$

$$\vec{w}\cdot A\vec{v} = \vec{w}^T A \vec{v} = \vec{v}^T A^T \vec{w} = \vec{v}^T A \vec{w} = \vec{v}\cdot A\vec{w} = \lambda_2 \vec{v}\cdot\vec{w}$$

$\Rightarrow \vec{v}\cdot\vec{w} = 0$

So if Gram-Schmidt E_0, get o.n. eigenbasis!
& normalize E_3 basis

$\vec{w}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ $\qquad \vec{w}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$\vec{z}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - -\frac{1}{\sqrt{2}}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix} \,/\!/\, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$\vec{w}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$ $\qquad S = \left[\vec{w}_1 \middle| \vec{w}_2 \middle| \vec{w}_3\right]$

$S^TAS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Bonus question: what (degenerate) quadric surface has points whose standard coordinates satisfy
$$x^2+y^2+z^2+2xy+2xz+2yz = 1$$

Another non-accident in page 1 example:

$\boxed{\text{All eigenvalues of symmetric real matrices are real}}$

proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of A, A real, symmetric
Let $\vec{v} \neq 0$ be a λ-eigenvector

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

write $\lambda = a+ib$
$\vec{v} = \vec{u} + i\vec{w}$

$$A(\vec{u}+i\vec{w}) = (a+ib)(\vec{u}+i\vec{w})$$

$$\Rightarrow \overline{A\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

$\bar{\ }$ is complex conjugation

$$\boxed{A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}}$$

So,
$$\bar{\vec{v}} \cdot A\vec{v} = \bar{\vec{v}} \cdot \lambda\vec{v} = (a+ib)(\|\vec{u}\|^2 + \|\vec{v}\|^2)$$
&
$$\vec{v} \cdot A\bar{\vec{v}} = \bar{\lambda}\,\vec{v}\cdot\bar{\vec{v}} = (a-ib)(\|u\|^2 + \|\vec{v}\|^2)$$

But $\bar{\vec{v}}^T A \vec{v} = (\quad)^T = v^T A \bar{\vec{v}}$

$$\bar{\vec{v}} \cdot A\vec{v} = \vec{v} \cdot A\bar{\vec{v}}$$

so $b = 0$!!

<u>Proof of spectral thm</u>

By induction!

- $n=1$; if $A = [a_{11}]$ then $A \cdot 1 = a_{11} 1$ so $\{1\}$ is an eigenbasis (already normalized too)
- Assume every $A_{(n-1)\times(n-1)}$ which is real and symmetric has an orthonormal eigenbasis

Let $A_{n\times n}$ real & symmetric
Let $\lambda_1 \in \mathbb{R}$ be an eigenvalue, $\vec{v} \neq \vec{0}$ $A\vec{v} = \lambda_1\vec{v}$.
Let $\vec{w}_1 = \frac{\vec{v}}{\|\vec{v}\|}$, complete to o.n. basis of $\mathbb{R}^n$, $\{\vec{w}_1, \vec{w}_2, \ldots \vec{w}_n\} = \mathcal{B}$

Let $T\vec{x} = A\vec{x}$, then
$$[T]_{\mathcal{B}} := B = \left[\begin{array}{c|c} \lambda_1 & ? \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right] = S^T A S$$

$$S = \left[\vec{w}_1 \middle| \vec{w}_2 \middle| \cdots \middle| \vec{w}_n\right]$$

<u>But</u> B is symmetric

since
$$\begin{aligned} b_{ij} &= \text{entry}_i\left([T\vec{w}_j]_{\mathcal{B}}\right) \\ &= \vec{w}_i \cdot T\vec{w}_j \quad (\text{because } \mathcal{B} \text{ is o.n.}) \\ &= \vec{w}_i \cdot A\vec{w}_j \\ &= \vec{w}_j \cdot A\vec{w}_i \\ &= \text{entry}_j(\text{col}_i\, B)) = b_{ji} \end{aligned}$$

So
$$B = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \tilde{B} \end{array}\right]$$

$\tilde{B}$ symmetric.

Since $\tilde{B}_{(n-1)\times(n-1)}$, inductive hypothesis says $\exists\, S_1$ (with cols an orthnormal eigenbasis of $\tilde{B}$)

$$S_1^T \tilde{B} S_1 = \text{diag} = \begin{bmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

So

$$\underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_1^T \end{array}\right]}_{S_2^T} \underbrace{\left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \widetilde{B} \end{array}\right]}_{\substack{B \\ \shortparallel \\ S^T A S}} \underbrace{\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_1 \end{array}\right]}_{S_2 \text{ orthog}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} := \Lambda$$

So $(SS_2)^T A \underbrace{(SS_2)}_{\text{orthog matrix}} = \Lambda$, i.e. the cols of SS_2 are an orthonormal eigenbasis for A $\blacksquare$

Application: 2^{nd} derivative test for $f(x,y)$ or $f(x,y,z)$ etc. at a critical point (x_0,y_0), (x_0,y_0,z_0) etc.

relies on Taylor expansion near $(x_0,y_0) = P$

$$f(x,y) = f(x_0,y_0) + [f_x(P) \quad f_y(P)] \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \underbrace{\tfrac{1}{2!} f_{xx}(P)(x-x_0)^2 + f_{xy}(P)(x-x_0)(y-y_0) + \tfrac{1}{2!} f_{yy}(P)(y-y_0)^2}_{\tfrac{1}{2!}[x-x_0,\, y-y_0] \begin{bmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}} + \text{higher order}$$

$$+ \text{ H.O}$$

$$f(x,y,z) = f(x_0,y_0,z_0) + [f_x(P), f_y(P), f_z(P)] \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} + \tfrac{1}{2!}[x-x_0, y-y_0, z-z_0] \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} + \text{H.O.}$$

Def $A_{n\times n}$ real, symm is

<u>positive definite</u> iff $A\vec{v}\cdot\vec{v} = \vec{v}^T A \vec{v} > 0 \quad \forall \vec{v} \neq \vec{0}$

<u>negative definite</u> iff $A\vec{v}\cdot\vec{v} < 0 \quad \forall \vec{v} \neq \vec{0}$

(evaluated at P)
"Hessian matrix" of 2^{nd} derivs is symmetric!
"$[D^2 f(P)]$

<u>2^{nd} deriv test</u>: If $\nabla f(P) = \vec{0}$ (i.e. all 1^{st} partials = 0)

Then if $[D^2 f(P)]$ is pos. def, P is local MIN (concave up in all dirs!)

if $[D^2 f(P)]$ is neg. def, P is local MAX (CD in all dirs!)

The 2^{nd} deriv. test is really an eigenvalues test!

$$[D^2 f(P)] = A \quad \text{is symmetric.}$$

Find S orthog matrix whose cols are eigenbasis for A

$$\vec{v}^T A \vec{v} = \vec{v}^T S^T \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} S \vec{v}$$

$$= \vec{u}^T \Big[\Lambda \Big] \vec{u} \qquad \vec{u} = S\vec{v}$$

$$= \sum_{i=1}^{n} \lambda_i u_i^2$$

so A is pos def iff all $\lambda_i > 0$

neg def iff all $\lambda_i < 0$

<u>examples</u>

- Does $f(x,y) = x^2 + y^2 + 4xy$ have a local ~~max~~ min at $(0,0)$?

  $\nabla f(0,0) = [0, 0]$

  $D^2 f(0,0) =$

  eigenvals =

- Does $f(x,y) = 2x^2 + y^2 + 3z^2 + xy - xz - yz$ have a local min at $(0,0)$?

  $\nabla f(0,0) = [0,0,0]$

  $$D^2 f(0,0,0) = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

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> A:=matrix(3,3,[4.,1,-1,1,2,-1,-1,-1,6]);
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$$A := \begin{bmatrix} 4. & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

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> eigenvalues(A);
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1.510711428, 3.710831454, 6.778457118

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>
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