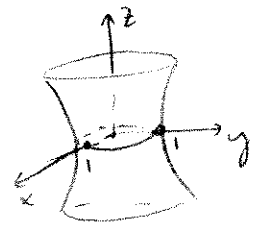
Math 2270-1
Fri Dec 2

Quadric surface zoo
(also, consult your old Calculus book)

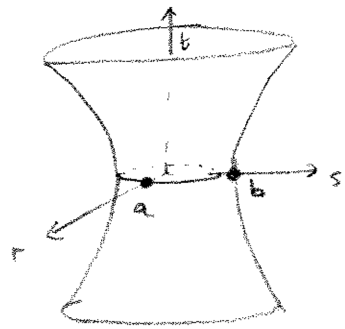
In the examples below I assume evals are ± 1 or 0, since space can be rescaled to make this so: e.g.

$x^2+y^2-z^2=1$ $\quad T\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}ax\\by\\cz\end{bmatrix}=\begin{bmatrix}r\\s\\t\end{bmatrix}$



$x^2+y^2-z^2=1$

iff $\frac{r^2}{a^2}+\frac{s^2}{b^2}-\frac{t^2}{c^2}=1.$



Zoo:

λ_1, λ_2 one sign
λ_3 opposite sign

cross sections $\perp$ to axis are ellipses ↓ — cross sections containing axis are hyperbolas ↓

$x^2+y^2-z^2=1$ (above) 1-sheeted elliptic hyperboloid

$x^2+y^2-z^2=0$ cone: 

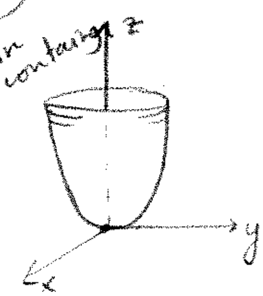
$x^2+y^2-z^2=-1$ ⟵ —— 2-sheeted elliptic hyperboloid

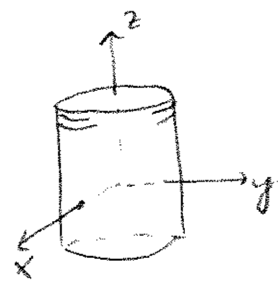
$\lambda_1, \lambda_2, \lambda_3$ same sign: ellipsoid

$x^2+y^2+z^2=1$ (sphere)
$x^2+y^2+z^2=0$ (point)
$x^2+y^2+z^2=-1$ (empty set)

$\lambda_3=0$, λ_1, λ_2 same sign — ellipse cross-sections $\perp$ axis ↓ ; parabolas in sections containing axis

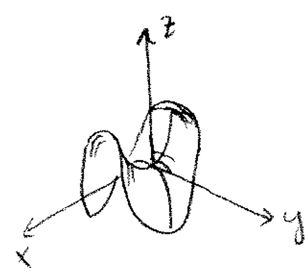
$z=x^2+y^2$ elliptic paraboloid
$1=x^2+y^2$ elliptic cylinder
$0=x^2+y^2$ vertical line
$-1=x^2+y^2$ $\emptyset$

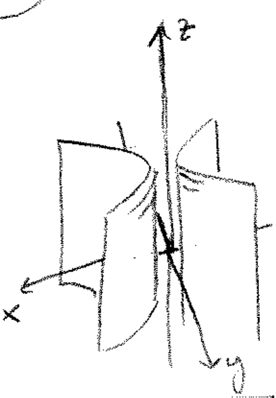




$\lambda_3=0$, λ_1, λ_2 opposite sign

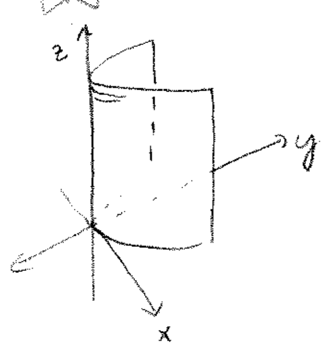
$z=x^2-y^2$ hyperbolic paraboloid ("saddle surface").
$1=x^2-y^2$ hyperbolic cylinder
$0=x^2-y^2$ crossing planes 

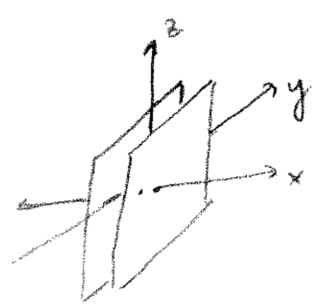




$\lambda_3=\lambda_2=0$, $\lambda_1\neq 0$

$y=x^2$ parabolic cylinder
$1=x^2$ parallel planes
$0=x^2$ single plane $(x=0)$
$-1=x^2$ $\emptyset$





$\lambda_1=\lambda_2=\lambda_3=0$: plane

Last homework assignment for 2270-1 !!

Due Fri 12/9 5 pm

[You may work in groups of up to 3 people]

- Kolman § 9.5 p. 423 (#25, 26, 27)

  for each problem find a positively oriented orthonormal basis $\mathcal{B} = \{\vec{w}_1, \vec{w}_2\}$

  so that the coordinates of points on the conic (wrt $\mathcal{B}$) satisfy a quadratic equation without cross terms. Exhibit this equation and draw the conic.

  Do <u>the algebra</u> in #25 by hand (but feel free to check with MAPLE!) All the rest of the work may be done with technology

- Kolman § 9.6 p 430-431 (15, 23, 25, 26)

  Find positively oriented orthonormal basis so that in the new coord system quadratic eqtn has no cross terms. Write new eqtn, identify quartic by name, sketch surface. Technology encouraged!

- Bretscher

  8.1 (#5, 14)

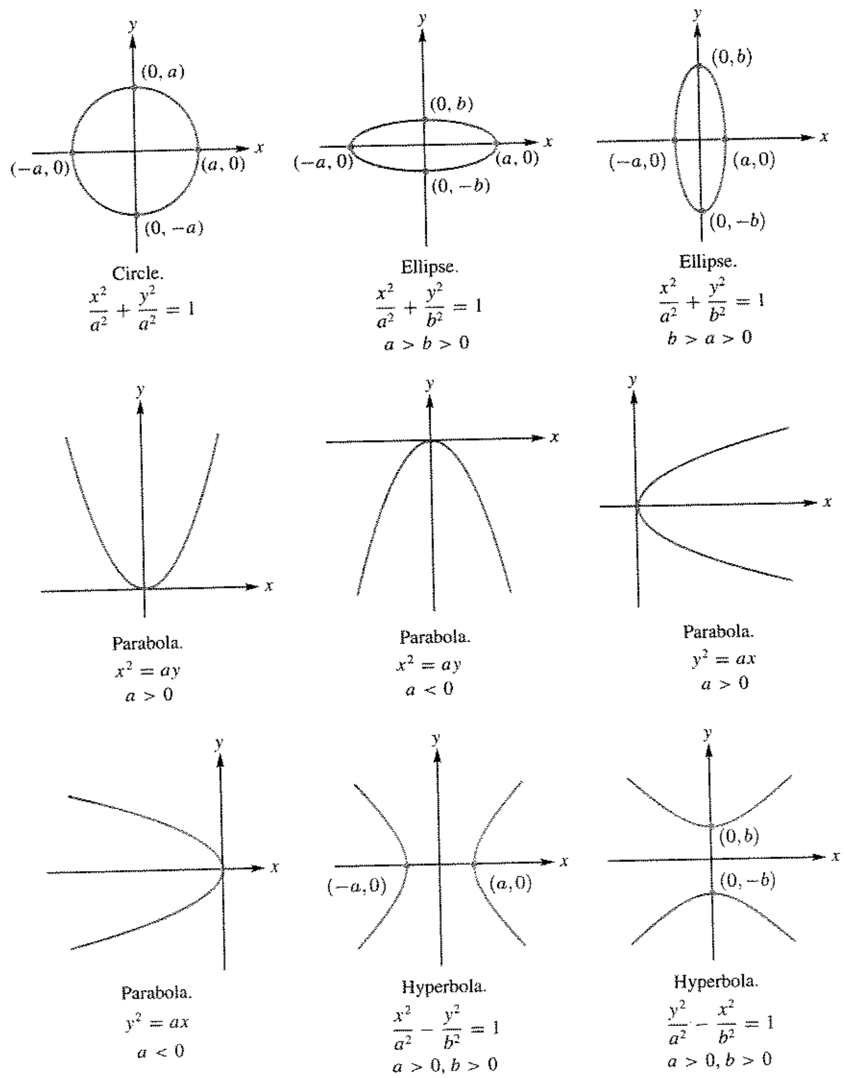
  8.2 (37)

  8.3 (7, 9, 10, 12)

If you do the bonus question in the Maple notes make sure you or your group hand this in separately to me

- precisely, email me your template so that I can check it!
- email the Maple file as an attachment.

Figure 9.18 ► The conic sections in standard position



Solution (a) We rewrite the given equation as

$$\frac{4}{100}x^2 + \frac{25}{100}y^2 = \frac{100}{100}$$

or

$$\frac{x^2}{25} + \frac{y^2}{4} = 1,$$

whose graph is an ellipse in standard position with $a = 5$ and $b = 2$. Thus the x-intercepts are $(5, 0)$ and $(-5, 0)$ and the y-intercepts are $(0, 2)$ and $(0, -2)$.

(b) Rewriting the given equation as

$$\frac{x^2}{9} - \frac{y^2}{4} = 1,$$

we see that its graph is a hyperbola in standard position with $a = 3$ and $b = 2$. The x-intercepts are $(3, 0)$ and $(-3, 0)$.

9.5 Exercises

In Exercises 1 through 10, identify the graph of the equation.

1. $x^2 + 9y^2 - 9 = 0.$

2. $x^2 = 2y.$

3. $25y^2 - 4x^2 = 100.$

4. $y^2 - 16 = 0.$

5. $3x^2 - y^2 = 0.$

6. $y = 0.$

7. $4x^2 + 4y^2 - 9 = 0.$

8. $-25x^2 + 9y^2 + 225 = 0.$

9. $4x^2 + y^2 = 0.$

10. $9x^2 + 4y^2 + 36 = 0.$

In Exercises 11 through 18, translate axes to identify the graph of the equation and write the equation in standard form.

11. $x^2 + 2y^2 - 4x - 4y + 4 = 0.$

12. $x^2 - y^2 + 4x - 6y - 9 = 0.$

13. $x^2 + y^2 - 8x - 6y = 0.$

14. $x^2 - 4x + 4y + 4 = 0.$

15. $y^2 - 4y = 0.$

16. $4x^2 + 5y^2 - 30y + 25 = 0.$

17. $x^2 + y^2 - 2x - 6y + 10 = 0.$

18. $2x^2 + y^2 - 12x - 4y + 24 = 0.$

In Exercises 19 through 24, rotate axes to identify the graph of the equation and write the equation in standard form.

19. $x^2 + xy + y^2 = 6.$

20. $xy = 1.$

21. $9x^2 + y^2 + 6xy = 4.$

22. $x^2 + y^2 + 4xy = 9.$

23. $4x^2 + 4y^2 - 10xy = 0.$

24. $9x^2 + 6y^2 + 4xy - 5 = 0.$

In Exercises 25 through 30, identify the graph of the equation and write the equation in standard form.

25. $9x^2 + y^2 + 6xy - 10\sqrt{10}\,x + 10\sqrt{10}\,y + 90 = 0.$

26. $5x^2 + 5y^2 - 6xy - 30\sqrt{2}\,x + 18\sqrt{2}\,y + 82 = 0.$

27. $5x^2 + 12xy - 12\sqrt{13}\,x = 36.$

28. $6x^2 + 9y^2 - 4xy - 4\sqrt{5}\,x - 18\sqrt{5}\,y = 5.$

29. $x^2 - y^2 + 2\sqrt{3}\,xy + 6x = 0.$

30. $8x^2 + 8y^2 - 16xy + 33\sqrt{2}\,x - 31\sqrt{2}\,y + 70 = 0.$

9.6 QUADRIC SURFACES

Prerequisite. Section 9.5, Conic Sections.

In Section 9.5 conic sections were used to provide geometric models for quadratic forms in two variables. In this section we investigate quadratic forms in three variables and use particular surfaces called quadric surfaces as geometric models. Quadric surfaces are often studied and sketched in analytic geometry and calculus. Here we use Theorems 9.2 and 9.3 to develop a classification scheme for quadric surfaces.

A **second-degree polynomial equation** in three variables x, y, and z has the form

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz = j, \qquad (1)$$

where coefficients a through j are real numbers with $a, b, \ldots, f$ not all zero. Equation (1) can be written in matrix form as

$$\mathbf{x}^T A\mathbf{x} + B\mathbf{x} = j, \qquad (2)$$

where

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}, \quad B = \begin{bmatrix} g & h & i \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We call $\mathbf{x}^T A\mathbf{x}$ the **quadratic form (in three variables) associated with the second-degree polynomial** in (1). As in Section 9.4, the symmetric matrix A is called the matrix of the quadratic form.

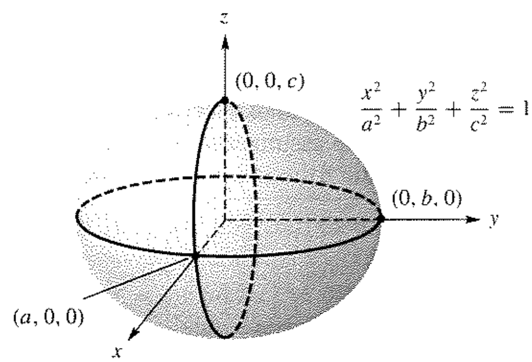


Figure 9.22 ▲
Ellipsoid

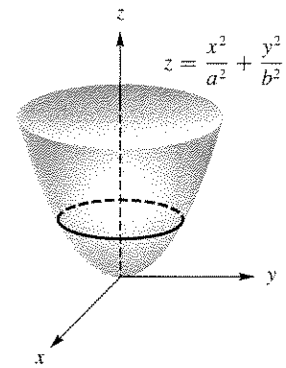


Figure 9.23 ▲
Elliptic paraboloid

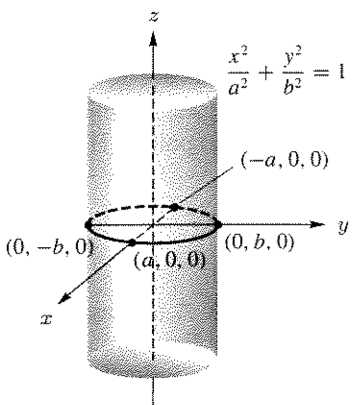


Figure 9.24 ▲
Elliptic cylinder

A degenerate case of a parabola is a line, so a degenerate case of an elliptic paraboloid is an **elliptic cylinder** (see Figure 9.24), which is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1, \qquad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Hyperboloid of One Sheet (See Figure 9.25.)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

A degenerate case of a hyperboloid is a pair of lines through the origin; hence a degenerate case of a hyperboloid of one sheet is a **cone** (Figure 9.26), which is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0.$$

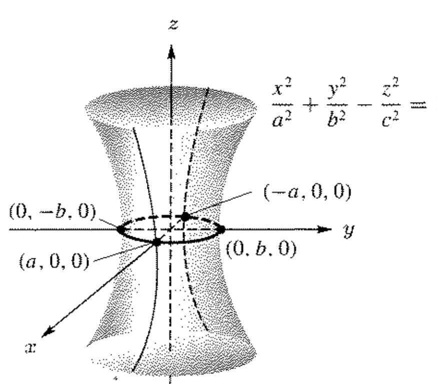


Figure 9.25 ▲
Hyperboloid of one sheet

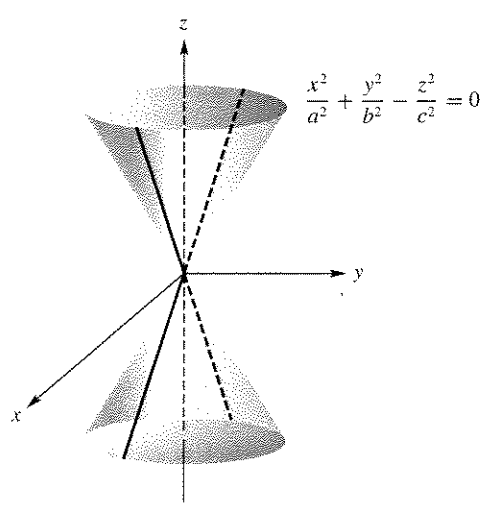


Figure 9.26 ▲
Cone

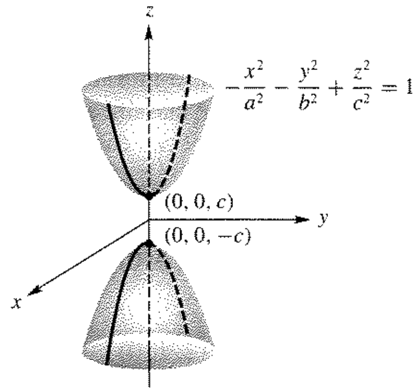


Figure 9.27 ▲
Hyperboloid of two sheets

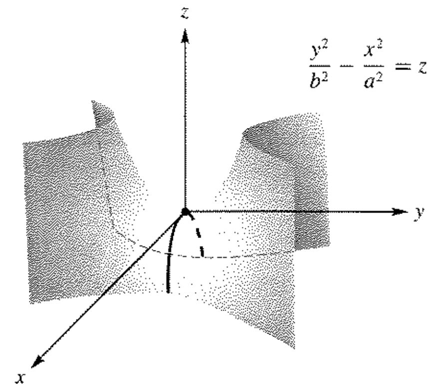


Figure 9.28 ▲
Hyperbolic paraboloid

Hyperboloid of Two Sheets (See Figure 9.27.)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hyperbolic Paraboloid (See Figure 9.28.)

$$\pm z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \qquad \pm y = \frac{x^2}{a^2} - \frac{z^2}{b^2}, \qquad \pm x = \frac{y^2}{a^2} - \frac{z^2}{b^2}.$$

A degenerate case of a parabola is a line, so a degenerate case of a hyperbolic paraboloid is a hyperbolic cylinder (see Figure 9.29), which is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1, \qquad \frac{x^2}{a^2} - \frac{z^2}{b^2} = \pm 1, \qquad \frac{y^2}{a^2} - \frac{z^2}{b^2} = \pm 1.$$

Parabolic Cylinder (See Figure 9.30.) One of a or b is not zero.

$$x^2 = ay + bz, \qquad y^2 = ax + bz, \qquad z^2 = ax + by.$$

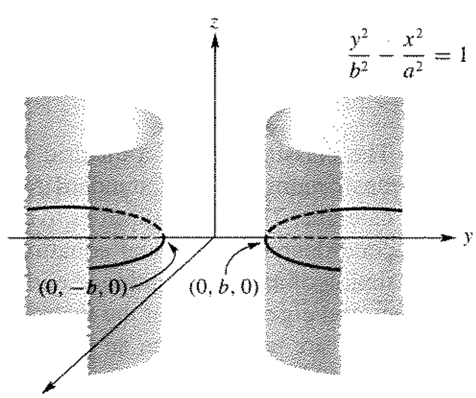


Figure 9.29 ▲
Hyperbolic cylinder

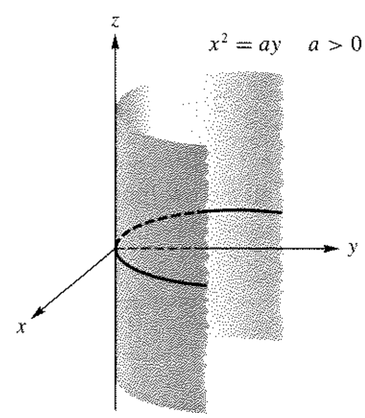


Figure 9.30 ▲
Parabolic cylinder

EXAMPLE 6 Continue with Example 5 to eliminate the first-degree terms.

Solution The last expression for the quadric surface in Example 5 can be written as

$$5x'^2 + \frac{9}{\sqrt{10}}x' + 2y'^2 - 5y' - 5z'^2 + \frac{3}{\sqrt{10}}z' = 2.$$

Completing the square in each variable, we have

$$5\left(x'^2 + \frac{9}{5\sqrt{10}}x' + \frac{81}{1000}\right) + 2\left(y'^2 - \frac{5}{2}y' + \frac{25}{16}\right)$$
$$- 5\left(z'^2 - \frac{3}{5\sqrt{10}}z' + \frac{9}{1000}\right)$$
$$= 5\left(x' + \frac{9}{10\sqrt{10}}\right)^2 + 2\left(y' - \frac{5}{4}\right)^2 - 5\left(z' - \frac{3}{10\sqrt{10}}\right)^2$$
$$= 2 + \frac{405}{1000} + \frac{50}{16} - \frac{45}{1000}.$$

Letting

$$x'' = x' + \frac{9}{10\sqrt{10}}, \qquad y'' = y' - \frac{5}{4}, \qquad z'' = z' - \frac{3}{10\sqrt{10}},$$

we can write the equation of the quadric surface as

$$5x''^2 + 2y''^2 - 5z''^2 = \frac{5485}{1000} = 5.485.$$

This can be written in standard form as

$$\frac{x''^2}{\frac{5.485}{5}} + \frac{y''^2}{\frac{5.485}{2}} - \frac{z''^2}{\frac{5.485}{5}} = 1. \quad \blacksquare$$

9.6 Exercises

In Exercises 1 through 14, use inertia to classify the quadric surface given by each equation.

1. $x^2 + y^2 + 2z^2 - 2xy - 4xz - 4yz + 4x = 8.$

2. $x^2 + 3y^2 + 2z^2 - 6x - 6y + 4z - 2 = 0.$

3. $z = 4xy.$

4. $x^2 + y^2 + z^2 + 2xy = 4.$

5. $x^2 - y = 0.$

6. $2xy + z = 0.$

7. $5y^2 + 20y + z - 23 = 0.$

8. $x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz = 16.$

9. $4x^2 + 9y^2 + z^2 + 8x - 18y - 4z - 19 = 0.$

10. $y^2 - z^2 - 9x - 4y + 8z - 12 = 0.$

11. $x^2 + 4y^2 + 4x + 16y - 16z - 4 = 0.$

12. $4x^2 - y^2 + z^2 - 16x + 8y - 6z + 5 = 0.$

13. $x^2 - 4z^2 - 4x + 8z = 0.$

14. $2x^2 + 2y^2 + 4z^2 + 2xy - 2xz - 2yz + 3x - 5y + z = 7.$

In Exercises 15 through 28, classify the quadric surface given by each equation and determine its standard form.

15. $x^2 + 2y^2 + 2z^2 + 2yz = 1.$

16. $x^2 + y^2 + 2z^2 - 2xy + 4xz + 4yz = 16.$

17. $2xz - 2z - 4y - 4z + 8 = 0$.

18. $x^2 + 3y^2 + 3z^2 - 4yz = 9$.

19. $x^2 + y^2 + z^2 + 2xy = 8$.

20. $-x^2 - y^2 - z^2 + 4xy + 4xz + 4yz = 3$.

21. $2x^2 + 2y^2 + 4z^2 - 4xy - 8xz - 8yz + 8x = 15$.

22. $4x^2 + 4y^2 + 8z^2 + 4xy - 4xz - 4yz + 6x - 10y + 2z = \frac{9}{4}$.

23. $2y^2 + 2z^2 + 4yz + \frac{16}{\sqrt{2}}x + 4 = 0$.

24. $x^2 + y^2 - 2z^2 + 2xy + 8xz + 8yz + 3x + z = 0$.

25. $-x^2 - y^2 - z^2 + 4xy + 4xz + 4yz + \frac{3}{\sqrt{2}}x - \frac{3}{\sqrt{2}}y = 6$.

26. $2x^2 + 3y^2 + 3z^2 - 2yz + 2x + \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = \frac{3}{8}$.

27. $x^2 + y^2 - z^2 - 2x - 4y - 4z + 1 = 0$.

28. $-8x^2 - 8y^2 + 10z^2 + 32xy - 4xz - 4yz = 24$.

Key Ideas for Review

- **Fibonacci sequence.**

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right].$$

- **Theorem 9.1.** If the $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively, then the general solution to the system of differential equations

$$\mathbf{x}' = A\mathbf{x}$$

is given by

$$\mathbf{x}(t) = b_1\mathbf{p}_1 e^{\lambda_1 t} + b_2\mathbf{p}_2 e^{\lambda_2 t} + \cdots + b_n\mathbf{p}_n e^{\lambda_n t}.$$

- **Theorem 9.2 (Principal Axes Theorem).** Any quadratic form in n variables $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is equivalent to a quadratic form,

$$h(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A of g.

- **Theorem 9.3.** A quadratic form $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ in n variables is equivalent to a quadratic form

$$h(\mathbf{y}) = y_1^2 + y_2^2 + \cdots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \cdots - y_r^2.$$

- The trajectories of the 2×2 dynamical system of the form

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

are completely determined by the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Supplementary Exercises

1. Let $A(t) = \left[a_{ij}(t)\right]$ be an $n \times n$ matrix whose entries are all functions of t; $A(t)$ is called a **matrix function**. The derivative and integral of $A(t)$ is defined componentwise; that is,

$$\frac{d}{dt}[A(t)] = \left[\frac{d}{dt}a_{ij}(t)\right]$$

and

$$\int_a^t A(s)\,ds = \left[\int_a^t a_{ij}(s)\,ds\right].$$

For each of the following matrices $A(t)$, compute $\frac{d}{dt}[A(t)]$ and $\int_0^t A(s)\,ds$.

(a) $A(t) = \begin{bmatrix} t^2 & \frac{1}{t+1} \\ 4 & e^{-t} \end{bmatrix}$.

(b) $A(t) = \begin{bmatrix} \sin 2t & 0 & 0 \\ 0 & 1 & -t \\ 0 & te^{t^2} & \frac{t}{t^2+1} \end{bmatrix}$.

2. For $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and each of the following matrices A, solve the initial value problem defined in Exercise T.6 below.

(a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.