MATH 2270-1
Symmetric matrices, conics and quadrics
Final homework problems

Kolman: section 9.5 problems 25, 26, 27
section 9.6 problems 15, 23, 25, 26

Conic sections: Section 9.5

```
> restart:with(linalg):with(plots):#for computations and pictures
  with(student):#to do algebra computations like completing the
  square
```

```
Warning, the protected names norm and trace have been redefined and unprotected
Warning, the name changecoords has been redefined
```

Using the template from December 2 Maple notes:

25)

```
> A:=matrix(2,2,[9,3,3,1]);
  B:=matrix(1,2,[-10*sqrt(10),10*sqrt(10)]);
  C:=90;
  fmat:=v->evalm(transpose(v)&*A&*v
                 + B&*v + C):
  f:=v->simplify(fmat(v)[1]):
  eigenvals(A); #show the eigenvalues
  data:=eigenvectors(A):
  v1:=data[1][3][1]:#first eigenvector
  v2:=data[2][3][1]:#second eigenvector
  w1:=v1/norm(v1,2):#normalized
  w2:=v2/norm(v2,2):#normalized
  P:=augment(w1,w2):#our orthogonal matrix,
    #unless we want to switch columns
    if det(P)<0 then
       P:=augment(w2,w1):
    fi:
  evalm(P);
  implicitplot(f([x,y])=0,x=-5..10,y=-5..5,
       grid=[100,100],color=‘black‘);
       #increase grid size for better picture, but too big
       #takes too long
  f(evalm(P&*[u,v])):#do the change of variables
  simplify(%):#simplify it!
  completesquare(%,u): #complete the square in u
  completesquare(%,v):#and v
  Eqtn:=%=0;
```

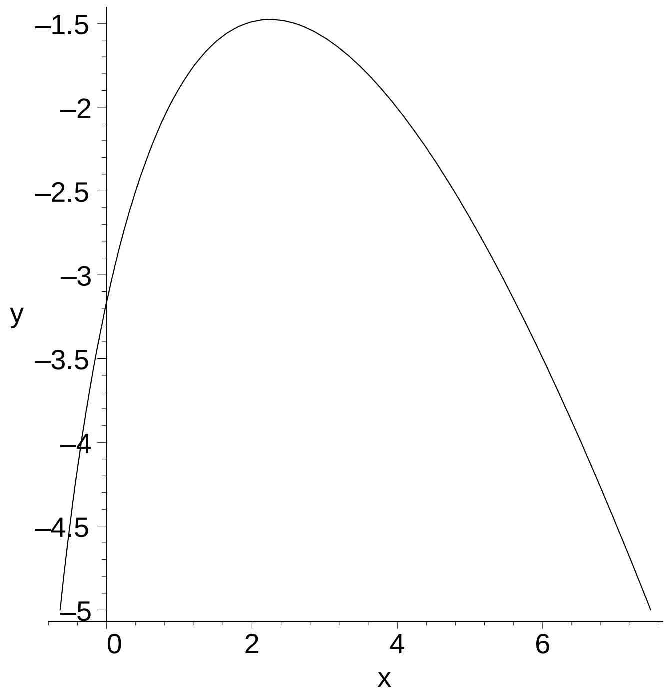
$$A := \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

$$B := \begin{bmatrix} -10\sqrt{10} & 10\sqrt{10} \end{bmatrix}$$

$$C := 90$$

$$0,\ 10$$

$$\begin{bmatrix} \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \\ -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{bmatrix}$$



$$Eqtn := 10\,(v-1)^2 + 80 - 40\,u = 0$$

This is a translated rotated parabola! The eigenvectors of the matrix A were 0,10, which meant we would get a parabola (or a degenerate conic). The positively oriented basis are the two columns of the exhibited transition matrix S above.

26)

```
> A:=matrix(2,2,[5,-3,-3,5]);
  B:=matrix(1,2,[-30*sqrt(2),18*sqrt(2)]);
  C:=82;
  fmat:=v->evalm(transpose(v)&*A&*v
                  + B&*v + C):
  f:=v->simplify(fmat(v)[1]):
  eigenvals(A); #show the eigenvalues
  data:=eigenvectors(A):
  v1:=data[1][3][1]:#first eigenvector
  v2:=data[2][3][1]:#second eigenvector
  w1:=v1/norm(v1,2):#normalized
  w2:=v2/norm(v2,2):#normalized
```

```
P:=augment(w1,w2):#our orthogonal matrix,
  #unless we want to switch columns
  if det(P)<0 then
     P:=augment(w2,w1):
  fi:
evalm(P);
implicitplot(f([x,y])=0,x=0..10,y=-5..5,
     grid=[100,100],color='black');
     #increase grid size for better picture, but too big
     #takes too long
f(evalm(P&*[u,v])):#do the change of variables
simplify(%):#simplify it!
completesquare(%,u): #complete the square in u
completesquare(%,v):#and v
Eqtn:=%=0;
```

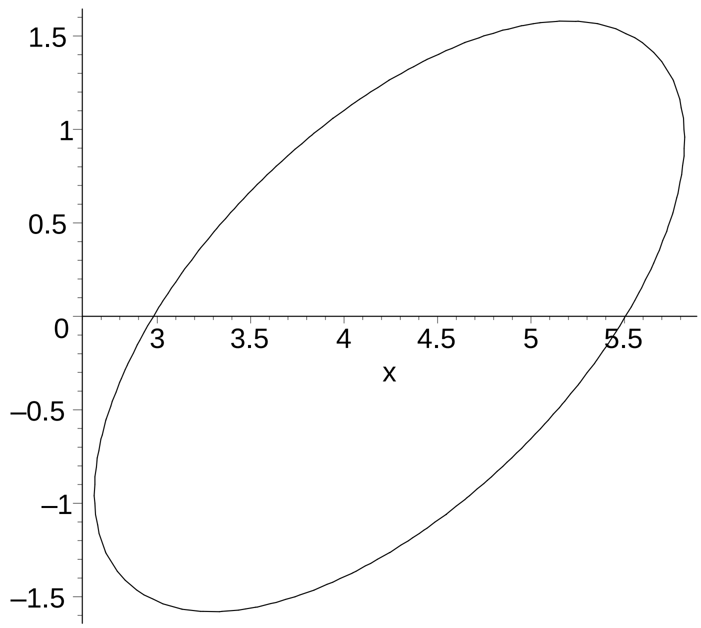
$$A := \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

$$B := [-30\sqrt{2} \quad 18\sqrt{2}]$$

$$C := 82$$

$$8, 2$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$



$$Eqtn := 8\,(v+3)^2 - 8 + 2\,(u-3)^2 = 0$$

This is a translated rotated ellipse! The eigenvectors of the matrix A were 2,8, which meant we would get an ellipse (or a degenerate conic). The positively oriented basis are the two columns of the exhibited transition matrix S above, and the center of the ellipse has coordinates u=3, v=-3 with respect to this basis.

27)

```
> A:=matrix(2,2,[5,6,6,0]);
  B:=matrix(1,2,[-12*sqrt(3),0]);
  C:=-36;
  fmat:=v->evalm(transpose(v)&*A&*v
                 + B&*v + C):
  f:=v->simplify(fmat(v)[1]):
  eigenvals(A); #show the eigenvalues
  data:=eigenvectors(A):
  v1:=data[1][3][1]:#first eigenvector
  v2:=data[2][3][1]:#second eigenvector
  w1:=v1/norm(v1,2):#normalized
  w2:=v2/norm(v2,2):#normalized
  P:=augment(w1,w2):#our orthogonal matrix,
    #unless we want to switch columns
    if det(P)<0 then
       P:=augment(w2,w1):
    fi:
  evalm(P);
  implicitplot(f([x,y])=0,x=-10..10,y=-10..10,
       grid=[100,100],color='black');
```

```
      #increase grid size for better picture, but too big
      #takes too long
 f(evalm(P&*[u,v])):#do the change of variables
 simplify(%):#simplify it!
 completesquare(%,u): #complete the square in u
 completesquare(%,v):#and v
 Eqtn:=%=0;
```

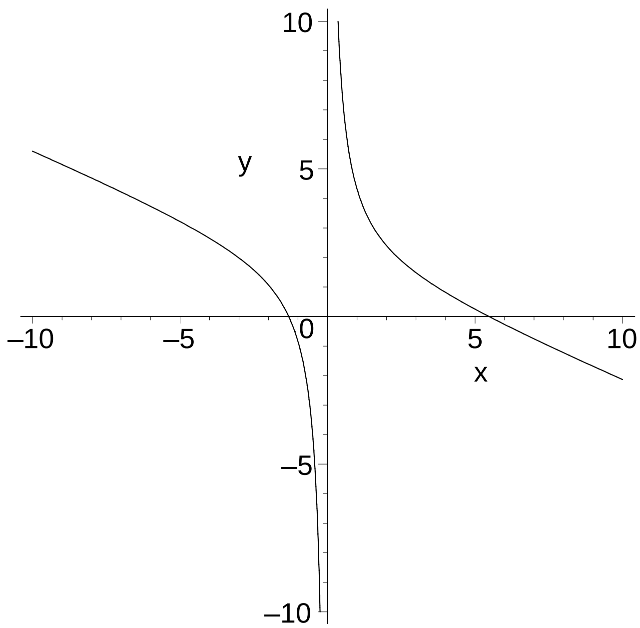
$$A := \begin{bmatrix} 5 & 6 \\ 6 & 0 \end{bmatrix}$$

$$B := [-12\sqrt{3} \quad 0]$$

$$C := -36$$

$$9, -4$$

$$\begin{bmatrix} \frac{2\sqrt{13}}{13} & \frac{3\sqrt{13}}{13} \\ -\frac{3\sqrt{13}}{13} & \frac{2\sqrt{13}}{13} \end{bmatrix}$$



$$Eqtn := 9\left(v - \frac{2\sqrt{3}\sqrt{13}}{13}\right)^2 - 36 - 4\left(u + \frac{3\sqrt{3}\sqrt{13}}{13}\right)^2 = 0$$

This is a translated rotated hyperbola! The eigenvectors of the matrix A were 9,-4 which meant we would get a hyperbola(or a degenerate conic). The positively oriented basis are the two columns of the exhibited transition matrix S above.

Quadric surfaces....I use a template I once wrote....
Kolman section 9.6, #15, 23, 25, 26
15)

```
> A:=matrix(3,3,[1,0,0,0,2,1,0,1,2]);
  B:=matrix([[0,0,0]]);
  C:=-1;#the constant term if the rhs is zero
  fmat:=v->evalm(transpose(v)&*A&*v
                  + B&*v + C):
  f:=v->fmat(v)[1]:
  simplify(f(vector([x,y,z])));
```

$$A := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$B := [0 \quad 0 \quad 0]$$

$$C := -1$$

$$-1 + x^2 + 2\,y^2 + 2\,y\,z + 2\,z^2$$

```
> eigenvals(A):
  data:=eigenvectors(A);#show the eigenvalues

    if data[2]=3  #the case of three equal eigenvalues;
                #quadratic part is already diagonalized
      then v1:=data[3][1];
           v2:=data[3][2];
           v3:=data[3][3];
     fi;

    if data[1][2]=2 #case of first eigenvalue has algebraic
               #multiplicity two, so second one has multiplicity one

     then  v1:=data[1][3][1];
           v2:=data[1][3][2];
           v3:=data[2][3][1];
      fi;

    if (data[1][2]=1 and  data[2][2]=2)
              #first eigenvalue mult. one, second
              #eigenvalue multiplicity two
      then
            v1:=data[1][3][1];
            v2:=data[2][3][1];
            v3:=data[2][3][2];
      fi;

    if (data[1][2]=1 and data[2][2]=1)
```

```
            #three distinct eigenvalues
      then
           v1:=data[1][3][1];
           v2:=data[2][3][1];
           v3:=data[3][3][1];
     fi;
```

$$data := [1, 2, \{[0, 1, -1], [1, 0, 0]\}], [3, 1, \{[0, 1, 1]\}]$$

$$v1 := [0, 1, -1]$$

$$v2 := [1, 0, 0]$$

$$v3 := [0, 1, 1]$$

Since all eigenvalues are positive I expect an ellipsoid (or it could degenerate to a point or nothing).

```
> almostP:=GramSchmidt([v1,v2,v3]);
  normP:=map(normalize,almostP);
  P:=augment(normP[1],normP[2],normP[3]);
```

$$almostP := [[0, 1, -1], [1, 0, 0], [0, 1, 1]]$$

$$normP := \left[\left[0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right], [1, 0, 0], \left[0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]\right]$$

$$P := \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> f(evalm(P&*[u,v,w])):#do the change of variables
  simplify(%);#simplify it!
  completesquare(%,u); #complete the square in u
  completesquare(%,v);#and v
  completesquare(%,w);#and w
  Eqtn:=%=0;
```

$$-1 + v^2 + u^2 + 3\, w^2$$

$$-1 + v^2 + u^2 + 3\, w^2$$

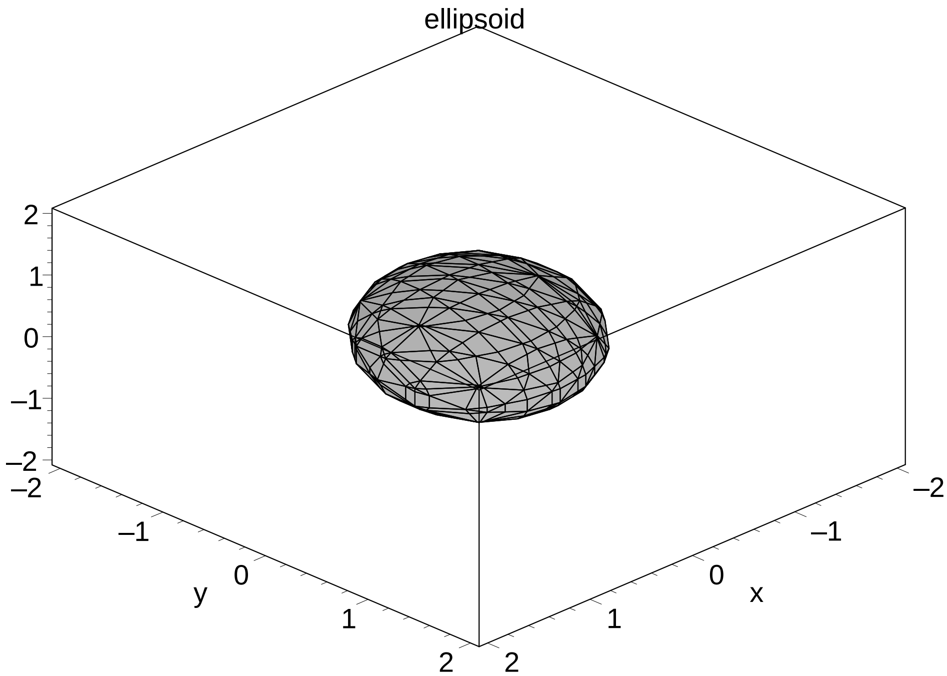
$$-1 + v^2 + u^2 + 3\, w^2$$

$$-1 + v^2 + u^2 + 3\, w^2$$

$$Eqtn := -1 + v^2 + u^2 + 3\, w^2 = 0$$

yup, ellipsoid

```
> implicitplot3d(f([x,y,z])=0,x=-2..2,y=-2..2,z=-2..2,
       grid=[15,15,15],axes=boxed,title=‘ellipsoid‘);
       #increase grid size for better picture, but too big
       #takes too long
```



#23)

```
> A:=matrix(3,3,[0,0,0,0,2,2,0,2,2]);
  B:=matrix([[16/sqrt(2),0,0]]);
  C:=4;#the constant term if the rhs is zero
  fmat:=v->evalm(transpose(v)&*A&*v
                 + B&*v + C):
  f:=v->fmat(v)[1]:
  simplify(f(vector([x,y,z])));
```

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$B := [8\sqrt{2} \quad 0 \quad 0]$$

$$C := 4$$

$$8\sqrt{2}\,x + 4 + 2\,y^2 + 4\,y\,z + 2\,z^2$$

```
> eigenvals(A):
  data:=eigenvectors(A);#show the eigenvalues

    if data[2]=3  #the case of three equal eigenvalues;
                #quadratic part is already diagonalized
      then v1:=data[3][1];
           v2:=data[3][2];
           v3:=data[3][3];
     fi;
```

```
    if data[1][2]=2 #case of first eigenvalue has algebraic
               #multiplicity two, so second one has multiplicity one

     then  v1:=data[1][3][1];
           v2:=data[1][3][2];
           v3:=data[2][3][1];
      fi;

    if (data[1][2]=1 and  data[2][2]=2)
              #first eigenvalue mult. one, second
              #eigenvalue multiplicity two
      then
            v1:=data[1][3][1];
            v2:=data[2][3][1];
            v3:=data[2][3][2];
      fi;

    if (data[1][2]=1 and data[2][2]=1)
            #three distinct eigenvalues
      then
           v1:=data[1][3][1];
           v2:=data[2][3][1];
           v3:=data[3][3][1];
     fi;
```

$$data := [4, 1, \{[0, 1, 1]\}], [0, 2, \{[0, 1, -1], [1, 0, 0]\}]$$

$$v1 := [0, 1, 1]$$

$$v2 := [1, 0, 0]$$

$$v3 := [0, 1, -1]$$

with one non-zero eigenvalue and two zero ones I expect a parabolic cylinder!

```
> almostP:=GramSchmidt([v1,v2,v3]);
  normP:=map(normalize,almostP);
  P:=augment(normP[1],normP[2],normP[3]);
```

$$almostP := [[0, 1, 1], [1, 0, 0], [0, 1, -1]]$$

$$normP := \left[\left[0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right], [1, 0, 0], \left[0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right]\right]$$

$$P := \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> f(evalm(P&*[u,v,w])):#do the change of variables
  simplify(%);#simplify it!
```

```
completesquare(%,u); #complete the square in u
completesquare(%,v);#and v
completesquare(%,w);#and w
Eqtn:=%=0;
```

$$8\sqrt{2}\,v+4+4\,u^2$$

$$8\sqrt{2}\,v+4+4\,u^2$$

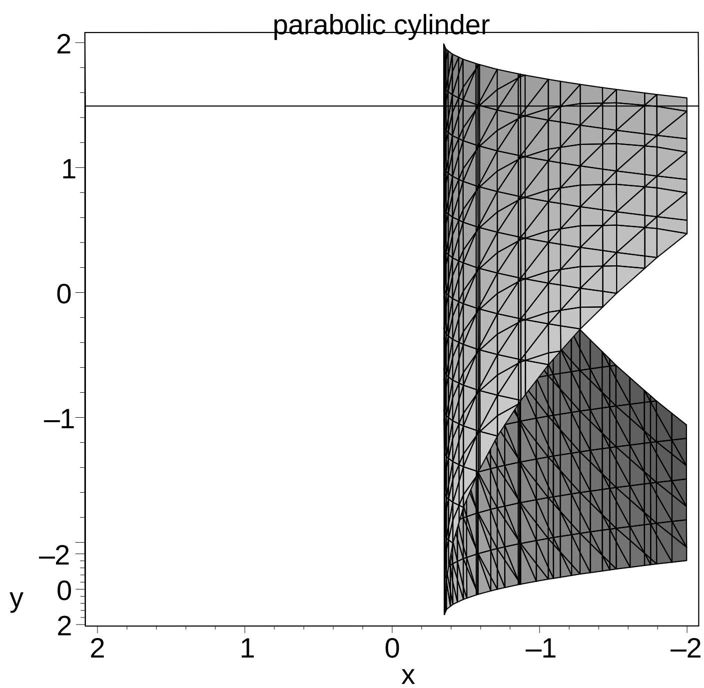
$$8\sqrt{2}\,v+4+4\,u^2$$

$$8\sqrt{2}\,v+4+4\,u^2$$

$$Eqtn := 8\sqrt{2}\,v+4+4\,u^2=0$$

yup, parabolic cylinder, opening in the negative v direction, cylinder in the w direction

```
> implicitplot3d(f([x,y,z])=0,x=-2..2,y=-2..2,z=-2..2,
      grid=[15,15,15],axes=boxed,title=‘parabolic cylinder‘);
      #increase grid size for better picture, but too big
      #takes too long
```



25)

```
> A:=matrix(3,3,[-1,2,2,2,-1,2,2,2,-1]);
  B:=matrix([[3/sqrt(2),-3/sqrt(2),0]]);
  C:=-6;#the constant term if the rhs is zero
  fmat:=v->evalm(transpose(v)&*A&*v
                     + B&*v + C):
  f:=v->fmat(v)[1]:
  simplify(f(vector([x,y,z])));
```

$$A := \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$B := \begin{bmatrix} \frac{3\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$C := -6$$

$$\frac{3}{2}\sqrt{2}\,x - \frac{3}{2}\sqrt{2}\,y - 6 - x^2 + 4\,x\,y + 4\,x\,z - y^2 + 4\,y\,z - z^2$$

```
> eigenvals(A):
  data:=eigenvectors(A);#show the eigenvalues

    if data[2]=3  #the case of three equal eigenvalues;
                #quadratic part is already diagonalized
      then v1:=data[3][1];
           v2:=data[3][2];
           v3:=data[3][3];
     fi;

    if data[1][2]=2 #case of first eigenvalue has algebraic
               #multiplicity two, so second one has multiplicity one

     then  v1:=data[1][3][1];
           v2:=data[1][3][2];
           v3:=data[2][3][1];
      fi;

    if (data[1][2]=1 and  data[2][2]=2)
              #first eigenvalue mult. one, second
              #eigenvalue multiplicity two
      then
            v1:=data[1][3][1];
            v2:=data[2][3][1];
            v3:=data[2][3][2];
      fi;

    if (data[1][2]=1 and data[2][2]=1)
            #three distinct eigenvalues
      then
           v1:=data[1][3][1];
           v2:=data[2][3][1];
           v3:=data[3][3][1];
     fi;
```

$$data := [-3, 2, \{[1, 0, -1], [0, 1, -1]\}], [3, 1, \{[1, 1, 1]\}]$$

$$v1 := [0, 1, -1]$$

$$v2 := [1, 0, -1]$$

$$v3 := [1, 1, 1]$$

with two evals of one sign, and one of the opposite, I shall get either a one or two sheeted hyperboloid, or a degnerate quadric

```
> almostP:=GramSchmidt([v1,v2,v3]);
  normP:=map(normalize,almostP);
  P:=augment(normP[1],normP[2],normP[3]);
```

$$almostP := \left[[0, 1, -1], \left[1, \frac{-1}{2}, \frac{-1}{2}\right], [1, 1, 1]\right]$$

$$normP := \left[\left[0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right], \left[\frac{\sqrt{3}\sqrt{2}}{3}, -\frac{\sqrt{3}\sqrt{2}}{6}, -\frac{\sqrt{3}\sqrt{2}}{6}\right], \left[\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]\right]$$

$$P := \begin{bmatrix} 0 & \frac{\sqrt{3}\sqrt{2}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}\sqrt{2}}{6} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}\sqrt{2}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix}$$

```
> f(evalm(P&*[u,v,w])):#do the change of variables
  simplify(%);#simplify it!
  completesquare(%,u); #complete the square in u
  completesquare(%,v);#and v
  completesquare(%,w);#and w
  Eqtn:=%=0;
```

$$\frac{3}{2}\sqrt{3}\, v - \frac{3}{2} u - 6 - 3 v^2 + 3 w^2 - 3 u^2$$

$$-3\left(u + \frac{1}{4}\right)^2 - \frac{93}{16} + \frac{3\sqrt{3}\, v}{2} - 3 v^2 + 3 w^2$$

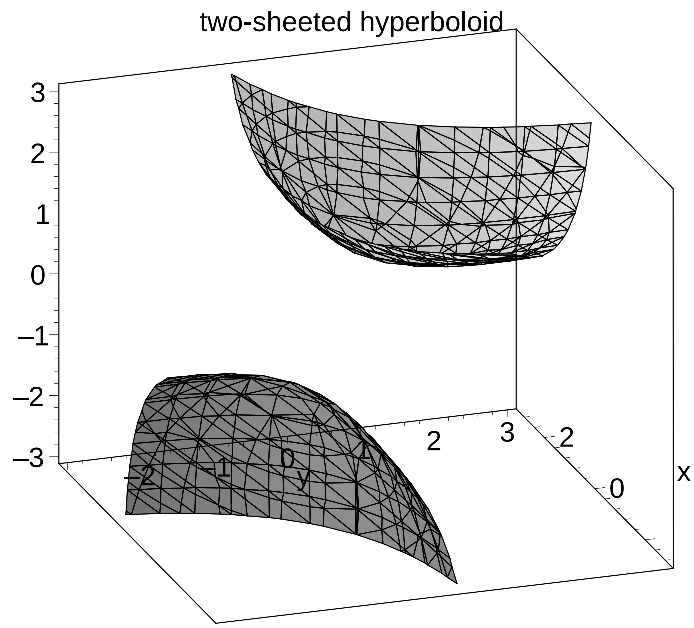
$$-3\left(v - \frac{\sqrt{3}}{4}\right)^2 - \frac{21}{4} - 3\left(u + \frac{1}{4}\right)^2 + 3 w^2$$

$$-3\left(v - \frac{\sqrt{3}}{4}\right)^2 - \frac{21}{4} - 3\left(u + \frac{1}{4}\right)^2 + 3 w^2$$

$$Eqtn := -3\left(v - \frac{\sqrt{3}}{4}\right)^2 - \frac{21}{4} - 3\left(u + \frac{1}{4}\right)^2 + 3 w^2 = 0$$

yup, two-sheeted hyerpboloid

```
> implicitplot3d(f([x,y,z])=0,x=-3..3,y=-3..3,z=-3..3,
       grid=[15,15,15],axes=boxed,title=‘two-sheeted hyperboloid‘);
       #increase grid size for better picture, but too big
       #takes too long
```



26

```
> A:=matrix(3,3,[2,0,0,0,3,-1,0,-1,3]);
  B:=matrix([[2,1/sqrt(2),1/sqrt(2)]]);
  C:=-3/8;#the constant term if the rhs is zero
  fmat:=v->evalm(transpose(v)&*A&*v
                  + B&*v + C):
  f:=v->fmat(v)[1]:
  simplify(f(vector([x,y,z])));
```

$$A := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$B := \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$C := \frac{-3}{8}$$

$$2\,x + \frac{1}{2}\sqrt{2}\,y + \frac{1}{2}\sqrt{2}\,z - \frac{3}{8} + 2\,x^2 + 3\,y^2 - 2\,y\,z + 3\,z^2$$

```
> eigenvals(A):
  data:=eigenvectors(A);#show the eigenvalues

    if data[2]=3  #the case of three equal eigenvalues;
                #quadratic part is already diagonalized
      then v1:=data[3][1];
```

```
            v2:=data[3][2];
            v3:=data[3][3];
      fi;

     if data[1][2]=2 #case of first eigenvalue has algebraic
                       #multiplicity two, so second one has multiplicity one

      then  v1:=data[1][3][1];
            v2:=data[1][3][2];
            v3:=data[2][3][1];
       fi;

     if (data[1][2]=1 and  data[2][2]=2)
                      #first eigenvalue mult. one, second
                      #eigenvalue multiplicity two
       then
             v1:=data[1][3][1];
             v2:=data[2][3][1];
             v3:=data[2][3][2];
       fi;

     if (data[1][2]=1 and data[2][2]=1)
                #three distinct eigenvalues
       then
            v1:=data[1][3][1];
            v2:=data[2][3][1];
            v3:=data[3][3][1];
      fi;
```

$$data := [2, 2, \{[0, 1, 1], [1, 0, 0]\}], [4, 1, \{[0, -1, 1]\}]$$

$$v1 := [0, 1, 1]$$

$$v2 := [1, 0, 0]$$

$$v3 := [0, -1, 1]$$

with all evals the same sign I shall get either an ellipsoid, or a point or the empty set

```
> almostP:=GramSchmidt([v1,v2,v3]);
  normP:=map(normalize,almostP);
  P:=augment(normP[1],normP[2],normP[3]);
```

$$almostP := [[0, 1, 1], [1, 0, 0], [0, -1, 1]]$$

$$normP := \left[\left[0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right], [1, 0, 0], \left[0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]\right]$$

$$P := \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

```
> f(evalm(P&*[u,v,w])):#do the change of variables
  simplify(%);#simplify it!
  completesquare(%,u); #complete the square in u
  completesquare(%,v);#and v
  completesquare(%,w);#and w
  Eqtn:=%=0;
```

$$2\,v + u - \frac{3}{8} + 2\,v^2 + 2\,u^2 + 4\,w^2$$

$$2\left(u + \frac{1}{4}\right)^2 - \frac{1}{2} + 2\,v + 2\,v^2 + 4\,w^2$$

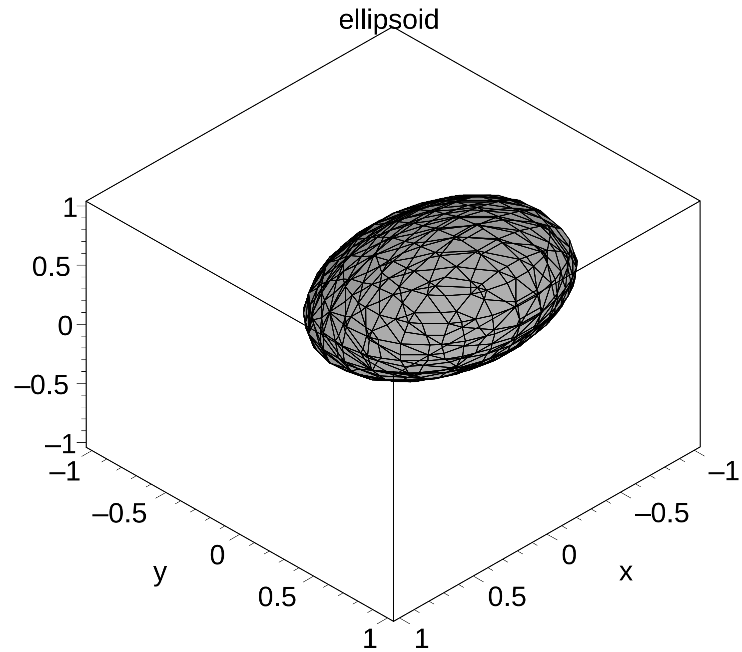
$$2\left(v + \frac{1}{2}\right)^2 - 1 + 2\left(u + \frac{1}{4}\right)^2 + 4\,w^2$$

$$2\left(v + \frac{1}{2}\right)^2 - 1 + 2\left(u + \frac{1}{4}\right)^2 + 4\,w^2$$

$$Eqtn := 2\left(v + \frac{1}{2}\right)^2 - 1 + 2\left(u + \frac{1}{4}\right)^2 + 4\,w^2 = 0$$

yup, ellipsoid

```
> implicitplot3d(f([x,y,z])=0,x=-1..1,y=-1..1,z=-1..1,
       grid=[15,15,15],axes=boxed,title=‘ellipsoid‘);
       #increase grid size for better picture, but too big
       #takes too long
```


ellipsoid
1
0.5
0
–0.5
–1
y
–1
–0.5
0
0.5
1
x
–1
–0.5
0
0.5
1

>