Math 2270-1

Wed 31 Aug

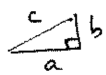
Reminder: HW due Friday at start of class

(optional) problem session Thurs (tomorrow), LCB 121 9:40-10:30

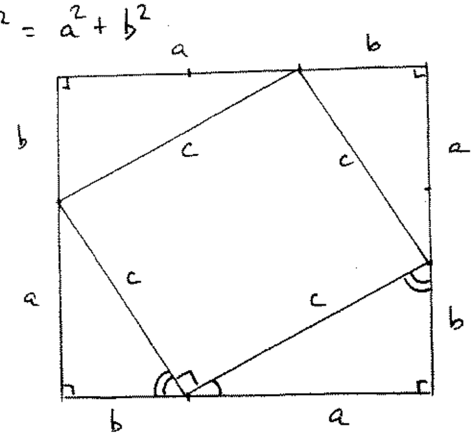
$\underbrace{\text{Appendix A}}_{\text{linear geometry review for } \mathbb{R}^2, \mathbb{R}^3}$ & $\underbrace{\text{Begin Chapter 2}}_{\text{Linear (matrix) transformation functions}}$

$\downarrow$

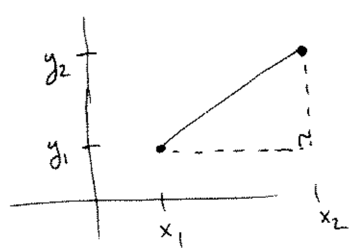
Pythagorean Theorem

Use this tile diagram, equating two computations of the total area, to deduce Pythagorean Thm for right Δ's , i.e.

$$c^2 = a^2 + b^2$$

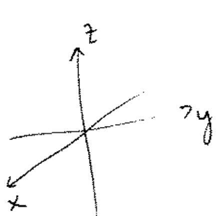


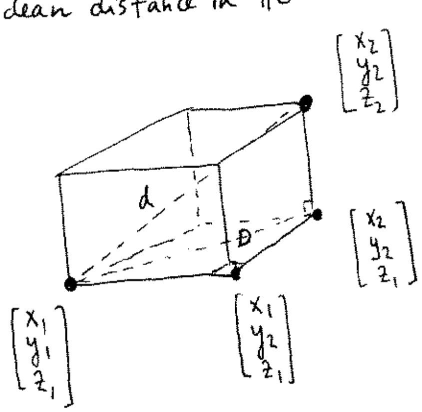
Corollary: Euclidean distance in $\mathbb{R}^2$



$$d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

Corollary: Euclidean distance in $\mathbb{R}^3$





$$d^2 = D^2 + (z_2-z_1)^2$$
$$= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Dot product and orthogonality

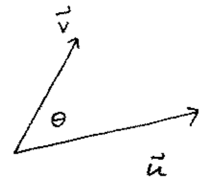
Recall from Tues, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $\vec{u}\cdot\vec{v} := \sum_{i=1}^{n} u_i v_i$

<u>Def</u>: $\|\vec{u}\| = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2} = \sqrt{\vec{u}\cdot\vec{u}}$ (= distance of geometric displacement $\vec{u}$)

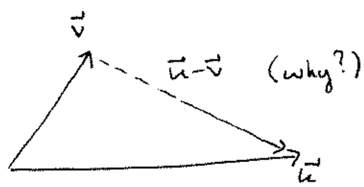
"magnitude" (length) of $\vec{u}$

<u>Theorem</u>: In $\mathbb{R}^2$ & $\mathbb{R}^3$ (and later $\mathbb{R}^n$)

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$, where $\measuredangle \vec{u},\vec{v} = \theta$:



<u>proof</u>: (In case θ is an acute angle ... other case analogous)
Assume $\|\vec{v}\| \le \|\vec{u}\|$ too; else relabel



compute $\|\vec{u}-\vec{v}\|^2$ two ways:

① $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

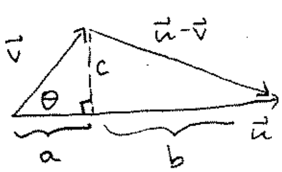
$= \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$

(<u>Why?!</u>: p438 A.5, can you check these algebra facts for dot prod?

1. $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
2. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
3. $(k\vec{v})\cdot\vec{w} = k(\vec{v}\cdot\vec{w}) = \vec{v}\cdot(k\vec{w})$
4. $\vec{v}\cdot\vec{v} > 0$ for $\vec{v} \ne \vec{0}$.

② 2 Pythag. applications



$\|\vec{u}-\vec{v}\|^2 = b^2 + c^2$

... finish & deduce result!

<u>Cor</u> $\vec{u}\cdot\vec{v} = 0$ iff $\theta = \pi/2$, i.e. $\vec{u} \perp \vec{v}$

is perpendicular to

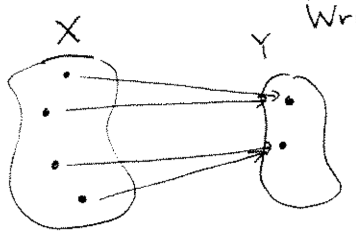
<u>Cor</u> $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$ (At least in $\mathbb{R}^2$ & $\mathbb{R}^3$ ~ discuss $\mathbb{R}^n$ truth in chapter 5).

For more review see Appendix A.

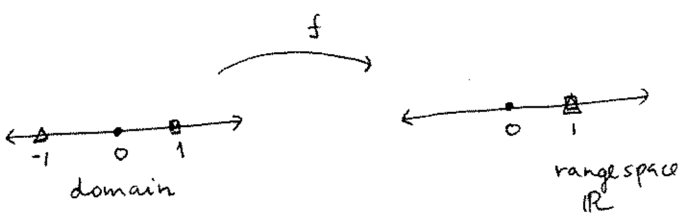
<u>Matrix (Linear) Transformations</u> §2.1

$f: X \to Y$, a <u>function</u> f is a rule (or way), which for each $x \in X$ assigns a particular $y \in Y$

Write $y = f(x)$



Calculus: $f: \mathbb{R} \to \mathbb{R}$ e.g $f(x) = x^2$

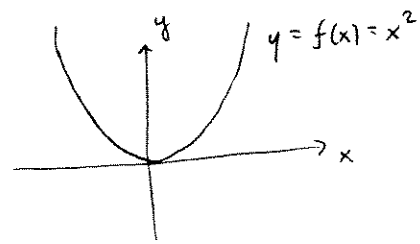


transformation picture

usually prefer this for $f: \mathbb{R} \to \mathbb{R}^n$, $n \geq 2$ (at least the range space part which shows a curve!)

or

$f: \mathbb{R}^n \to \mathbb{R}^m$ when $n+m \geq 3$ and the entire graph becomes unseeable to mere mortals!

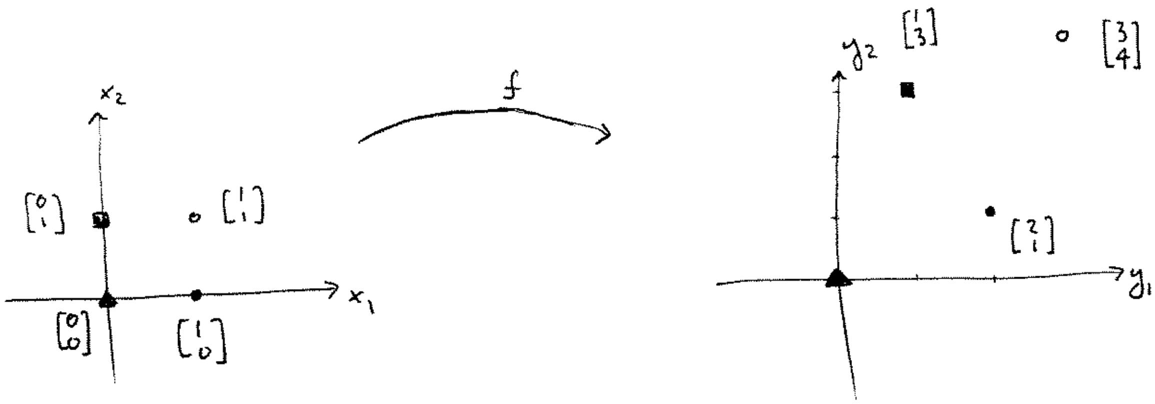


graph

usually prefer this for $f: \mathbb{R} \to \mathbb{R}$ or $f: \mathbb{R}^2 \to \mathbb{R}$

<u>Example</u> (See also the nice story example page 41).

$$\boxed{f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}$$



- how can we fill in the rest of the picture?
- Does the transformation f have an inverse transformation? If so, how can we find it?

what is the image of the x_1-axis, i.e. what does it get transformed to?

$$\text{x_1-axis} = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ s.t. } t \in \mathbb{R} \right\}$$

$$f\left(\begin{bmatrix} t \\ 0 \end{bmatrix}\right) =$$

where does the x_2 axis transform to?

General properties:

Let $\boxed{f(\vec{x}) = A\vec{x}}$ $\quad \vec{x} \in \mathbb{R}^n, \ A_{m\times n}, \ f(\vec{x}) \in \mathbb{R}^m$

be any matrix transformation.

Then

$$f(\vec{u}+\vec{v}) = (u_1+v_1)\,col_1 A + (u_2+v_2)\,col_2(A) + \dots + (u_n+v_n)\,col_n(A)$$
$$= u_1 col_1(A) + \dots + u_n col_n(A) \quad + \quad v_1 col_1(A) + \dots + v_n col_n(A)$$

also, $\left\{ \boxed{\begin{array}{l} f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v}) \\ f(k\vec{u}) = k\, f(\vec{u}) \end{array}} \right.$

Now, any line L in the domain can be represented parametrically with position vectors:

$$L = \{ \vec{u} + t\underset{\substack{\uparrow \\ \text{dir. or} \\ \text{vel. vector}}}{\vec{v}} \ \text{ s.t. } t \in \mathbb{R} \}$$

The image (transformation) of L, written as $f(L)$,

is $\{ f(\vec{u} + t\vec{v}) \ \text{ s.t. } t \in \mathbb{R} \}$

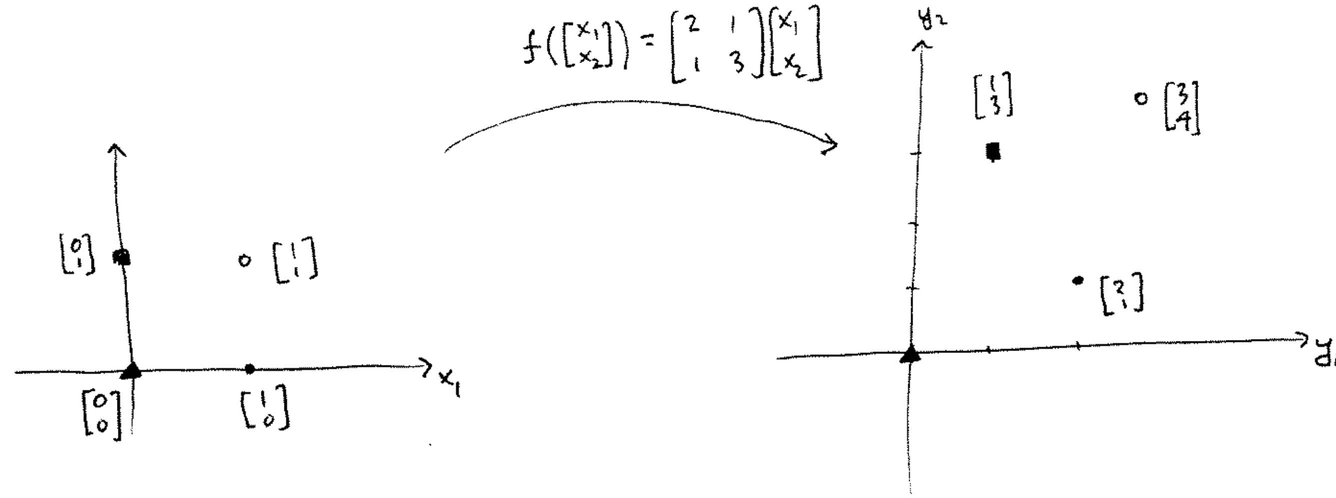
$= \{ f(\vec{u}) + t f(\vec{v}) \ \text{ s.t. } t \in \mathbb{R} \}$ by properties above — is a line in the range space! (or a point if $f(\vec{v}) = \vec{0}$)

- lines $\rightarrow$ lines (or points)
- parallel lines $\rightarrow$ parallel lines
- If S is any domain set, and $S + \vec{b} = \{ \vec{v} + \vec{b} \ \text{ s.t. } \vec{v} \in S \}$ is a translation by $\vec{b}$ — then the transformation $f(S+\vec{b}) = f(S) + f(\vec{b})$ is a translation of $f(S)$
- If $kS = \{ k\vec{v} \ \text{ s.t. } \vec{v} \in S \}$ is a scaling of S, then $f(kS) = k\, f(S)$ is a commensurate scaling of $f(S)$.

Example page 4 cont'd

Fill in the transformation picture!

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Inverse transformation:

$$2x_1 + x_2 = y_1$$
$$x_1 + 3x_2 = y_2$$

$$\begin{array}{r|cc|l}
 & 2 & 1 & y_1 \\
 & 1 & 3 & y_2 \\
\hline
R_1 & 1 & 3 & y_2 \\
R_2 & 2 & 1 & y_1 \\
\hline
 & 1 & 3 & y_2 \\
-2R_1+R_2 & 0 & -5 & y_1 - 2y_2 \\
\hline
 & 1 & 3 & y_2 \\
R_2/-5 & 0 & 1 & -y_1/5 + \frac{2}{5}y_2 \\
\hline
-3R_2+R_1 & 1 & 0 & \frac{3}{5}y_1 - \frac{1}{5}y_2 \\
 & 0 & 1 & -\frac{y_1}{5} + \frac{2}{5}y_2
\end{array}$$

$$f^{-1}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

?!