Consider the linear system

\[
\begin{cases}
q_{11}x_1 + q_{12}x_2 + \ldots + q_{1n}x_n = b_1 \\
q_{21}x_1 + q_{22}x_2 + \ldots + q_{2n}x_n = b_2 \\
\vdots \\
q_{m1}x_1 + q_{m2}x_2 + \ldots + q_{mn}x_n = b_m
\end{cases}
\]

(\text{LS})

\[
A_{m \times n} := \begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{m1} & q_{m2} & \cdots & q_{mn}
\end{bmatrix}, \quad \quad b := \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
\]

\[
\text{We solve this system by computing the reduced row echelon form of } A \text{ augmented by } b \text{ (the "augmented" matrix)}
\]

\[
\text{rref} \left[\begin{array}{c|c}
q_{11} & \cdots & q_{1n} & b_1 \\
q_{21} & \cdots & q_{2n} & b_2 \\
\vdots & \ddots & \vdots & \vdots \\
q_{m1} & \cdots & q_{mn} & b_m
\end{array}\right] = \text{rref} \left(\begin{array}{c|c}
A & b
\end{array}\right)
\]

\[
\text{Let's think about the universe of possibilities for the solution set, in terms of:}
\]

- # rows (A) \quad (m)
- # cols (A) \quad (n)
- # non-zero rows in rref(A) \quad \text{called the rank of } A, \quad \text{rank}(A)

\text{(may need also consider rank}(A; b) \text{ in this discussion).}
Example:

\[
\begin{bmatrix}
3 & 6 & 7 & 2 & 5 & 10 \\
2 & 4 & 2 & 4 & 2 & 4 \\
2 & 4 & 3 & 3 & 3 & 4 \\
3 & 6 & 6 & 3 & 6 & 6
\end{bmatrix}
\quad \text{rref}\left[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 & 2 \\
0 & 0 & 1 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\right]
\]

- What is the general solution?
  (Call the unknowns \( x_1, x_2, \ldots, x_5 \))

- How many "free variables" appear in the general solution?
  (i.e. free parameters)

General Case Questions:
- If the system has no solutions it's called inconsistent. How is this reflected in \( \text{rref}(A|b) \)?
• What must \( \text{rref}(A; b) \) look like in order that the system have a unique (single) solution? Express this in terms of \( m, n \) and \( \text{rank}(A), \text{rank}(A; b) \).

• Under exactly what conditions will the system have infinitely many solutions? (in terms of \( m, n, \text{rank}(A), \text{rank}(A; b) \)).

• If the system is consistent, express the number of free variables in terms of \( m, n, \text{rank}(A) \).

Note: to test your reasoning ability further, check out T-F questions, page 38-40.
Matrix, vector algebra intro.

Recall, you know how to scalar multiply and add vectors:

\[
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot t = \begin{bmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_n \end{bmatrix}
\]

\[
\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}
\]

As well, recall that \(n\)-vectors can represent displacements in \(\mathbb{R}^n\)

(which is why we use the arrow notation \(\vec{v}\))

and that scalar multiplication of a vector yields (represents)

a parallel displacement, in the same dir if scalar \(>0\)

in opposite dir if scalar \(<0\)

and that the abs value of the scalar is the length-scaling factor

and that vector addition corresponds to net displacement after doing \(\vec{v}\) and \(\vec{w}\)

Example: Represent \(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \end{bmatrix}\) geometrically as displacement:
The two basic ways to interpret a linear system. (Actually, a matrix times a vector.)

You've been taught (probably), that Matrix times vector is computed by:

\[
\begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
q_{m1} & q_{m2} & \cdots & q_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
q_{11}x_1 + q_{12}x_2 + \cdots + q_{1n}x_n \\
q_{21}x_1 + q_{22}x_2 + \cdots + q_{2n}x_n \\
\vdots \\
q_{m1}x_1 + \cdots + q_{mn}x_n
\end{bmatrix}
= 
\begin{bmatrix}
\text{row}_1(A) \cdot \vec{x} \\
\text{row}_2(A) \cdot \vec{x} \\
\vdots \\
\text{row}_m(A) \cdot \vec{x}
\end{bmatrix}
\]

Thus we can write the page 1 (LS) more compactly as

\[(LS) \quad A\vec{x} = \vec{b}\]

Notice from the intermediate equality above, that we can also interpret \(A\vec{x}\) as a linear combination of the columns of \(A\).

\[
A\vec{x} = x_1\text{col}_1(A) + x_2\text{col}_2(A) + \cdots + x_n\text{col}_n(A) = 
\begin{bmatrix}
\text{row}_1(A) \cdot \vec{x} \\
\text{row}_2(A) \cdot \vec{x} \\
\vdots \\
\text{row}_m(A) \cdot \vec{x}
\end{bmatrix}
\]

**Example:** Interpret \[
\begin{bmatrix}
3 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
11 \\
7
\end{bmatrix}
\]
as an intersecting line problem and as a linear combination problem.

Alg. ans is \[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
2
\end{bmatrix}
\]

Intersecting lines \(3x + y = 11\) \(x + 2y = 7\)

Linear combos \(x\begin{bmatrix}
3 \\
1
\end{bmatrix} + y\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
11 \\
7
\end{bmatrix}\)