Math 2270-1
Tues 30 Aug.

§1.3 Consider the linear system

$$(LS)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\underset{\substack{\uparrow \quad \nwarrow \\ \#\text{rows} \quad \#\text{cols}}}{A_{m\times n}} := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \qquad \vec{b} := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We solve this system by computing the reduced row echelon form of A augmented by b
(the "augmented" matrix)

$$\operatorname{rref}\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & & \vdots & \vdots \\ a_{m1} & & a_{mn} & b_m \end{array}\right] = \operatorname{rref}\left(A \mid \vec{b}\right)$$

Let's think about the universe of possibilities for the solution set, in terms of

rows (A) (m)
cols (A) (n)
non-zero rows in rref(A) $\longrightarrow$ called the <u>rank</u> of A, rank(A)
(maybe also consider rank(A|b) in this discussion.

Example :

$[A \,|\, \vec{b}]$ $\qquad$ rref $[A \,|\, b]$

$$\left[\begin{array}{ccccc|c} 3 & 6 & 7 & 2 & 5 & 10 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- What is the general solution?
  (Call the unknowns $x_1, x_2, \dots x_5$)

- How many "free variables" appear in the general solution?
  (i.e. free parameters)

<u>General Case Questions</u>:

- If the system has <u>no</u> sol'ns it's called inconsistent.
  How is this reflected in rref$(A \,|\, b)$?

(over)

- What must rref$(A \vdots b)$ look like in order that the system have a unique (single) solution? Express this in terms of m, n and rank(A), rank$(A \vdots b)$

- Under exactly what conditions will the system have ∞'ly many sol's? (in terms of m, n, rank(A), rank$(A|b)$)

- If the system is consistent, express the number of free variables in terms of m, n, rank(A).

Note: to test your reasoning ability further, check out T-F questions, page 38-40

Matrix, vector algebra intro.

Recall, you know how to scalar multiply and add vectors:

$$t\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} := \begin{bmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_n \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} := \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix}$$

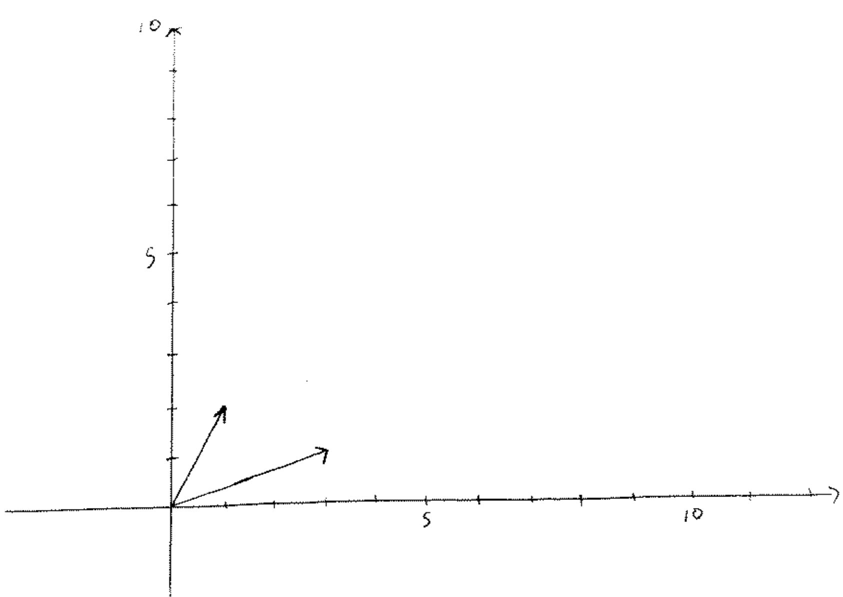
As well, recall that n-vectors can represent <u>displacements</u> in $\mathbb{R}^n$ (which is why we use the arrow notation $\vec{v}$)

and that <u>scalar multiplication</u> of a vector yields (represents) a parallel displacement, in the same dir if scalar > 0, in opposite dir if scalar < 0

and that the abs value of the scalar is the length-scaling factor

and that <u>vector addition</u> corresponds to net displacement after doing $\vec{v}$ and $\vec{w}$

<u>Example</u>: Represent $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}3\\1\end{bmatrix}$, $3\begin{bmatrix}3\\1\end{bmatrix}$, $2\begin{bmatrix}1\\2\end{bmatrix}$, $3\begin{bmatrix}3\\1\end{bmatrix}+2\begin{bmatrix}1\\2\end{bmatrix}$ geometrically as displacements:



The two basic ways to interpret a linear system. (Actually, a matrix times a vector)

You've been taught (probably), that Matrix times vector is computed by:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \text{row}_1(A)\cdot\vec{x} \\ \text{row}_2(A)\cdot\vec{x} \\ \vdots \\ \text{row}_m(A)\cdot\vec{x} \end{bmatrix}$$

Thus we can write the page 1 (LS) more compactly as

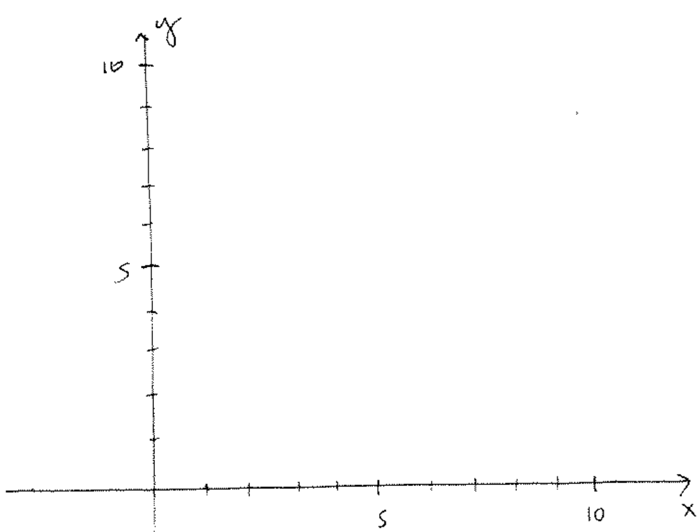
$$(LS) \qquad A\vec{x} = \vec{b}$$

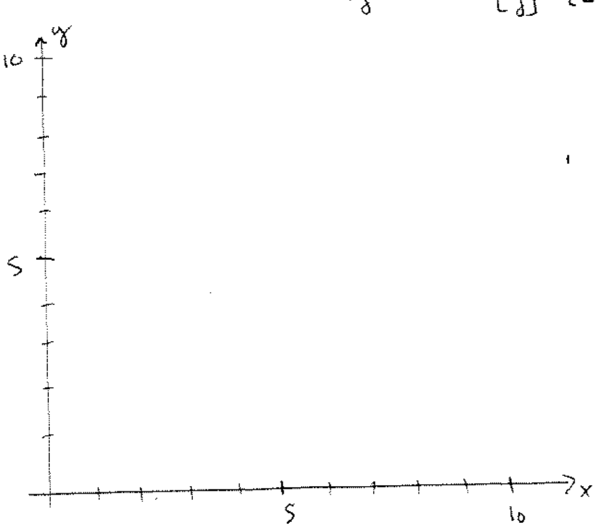
Notice from the intermediate equality above, that we can also interpret $A\vec{x}$ as a $\underbrace{\text{linear combination}}_{\text{sum of scalar multiples}}$ of the columns of A!

$$A\vec{x} = x_1\,\text{col}_1(A) + x_2\,\text{col}_2(A) + \cdots + x_n\,\text{col}_n(A) = \begin{bmatrix} \text{row}_1(A)\cdot\vec{x} \\ \text{row}_2(A)\cdot\vec{x} \\ \vdots \\ \text{row}_m(A)\cdot\vec{x} \end{bmatrix}$$

Example : Interpret $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$ as an intersecting line problem and as a linear combination problem

alg. ans is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$





intersecting lines $\begin{aligned} 3x + y &= 11 \\ x + 2y &= 7 \end{aligned}$

linear combos. $x\begin{bmatrix} 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$