On Friday Prof. Alfeld showed you a selection of (surprising) applications of the general linear system

\[
\begin{align*}
\begin{cases}
    a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
    a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
    \vdots \hfill & \vdots \\
    a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m
\end{cases}
\end{align*}
\]

- \(a_{ij}\) \text{ coefficients, known}
- \(b_i\) \text{ right hand sides, known}
- \(x_j\) \text{ unknowns}

Many of these, and also different examples will appear throughout the course. For example, this week's HW includes small examples of:

- polynomial interpolation
- Leontief input-output models
- chemical equation balancing
- physics momentum problems

Example: setup 31.1 #21.
Today: Gauss-Jordan elimination (did example Wed.)

> greatest mathematician ever?
> * algebra
> * probability
> * geometry

p. 19 story of Gauss used "least squares" solution to 17 linear eqns in 7 unknowns, (and invented) to calculate orbit of Ceres... (also p. 8, Chinese knew this algorithm for solving linear systems 2000 years earlier.)

Let's try solving (§1.2 #16)

\[
\begin{align*}
3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 &= 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 &= 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 &= 11
\end{align*}
\]

by hand (?!?) to understand how to reduce matrices, since "reduced row echelon form", the result of performing G-J elimination, is not only practically useful, but also key to understanding many "why's".

We may use the 3 elementary row operations
* multiply row by non-zero constant
* interchange 2 rows
* replace a row by its sum with a multiple of another row.
\[
\begin{array}{cccccc}
1 & 2 & 3 & 0 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Does each matrix have exactly one \text{rref}?  
\text{ans: yes; easier to show in chapter 3.}

\underline{Solution to original system:}

\underline{\text{rref}}  
\text{Reduced row echelon form: p.16}

a) If a row has any non-zero entries, then the first non-zero entry is a 1, called the \text{leading 1} of this row.

b) If a \underline{column} contains a \underline{row's} \text{leading 1}, all other column entries are \text{zero}.

c) As you move down the rows of the matrix, the \text{leading 1's move to the right by at least one column.}

What if \text{rref had been}

\[
\begin{array}{cccccc}
1 & 2 & 3 & 0 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 7 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]
In HW (4.1.2 #24,25) you ponder the fact that every elementary row operation can be reversed by an “inverse” row operation.

Since row ops correspond to equation operations, and since a solution \[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\] to a linear system is also a solution to a linear system obtained by doing an elementary column operation, we can deduce: elementary equation ops don’t change solution set.

So, the solution you read off from (correct) RREF is the solution to the original system.

**Geometric example**

\[
\begin{align*}
x + 3y &= 5 \\
2x - y &= 3
\end{align*}
\]

\[
\begin{bmatrix}
  1 & 3 & 5 \\
  2 & -1 & 3
\end{bmatrix}
\]

\[
\begin{align*}
x + 3y &= 5 \\
-7y &= -7
\end{align*}
\]

\[
\begin{bmatrix}
  1 & 3 & 5 \\
  -7 & 0 & -7
\end{bmatrix}
\]

\[
\begin{align*}
x &= 2 \\
y &= 1
\end{align*}
\]

\[
\begin{bmatrix}
  1 & 3 & 5 \\
  0 & 0 & 2
\end{bmatrix}
\]

\[
\begin{align*}
x &= 2 \\
y &= 1
\end{align*}
\]
Let Maple compute rref for you, in big examples!

```maple
> with(linalg): # load linear algebra library of procedures
M:=matrix([0,0,1,-1,4,2,4,2,4,4,3,3,4,3,6,6,3,6,6]);
M:=
| 0  0  1 -1  4 |
| 2  4  2  4  2 |
| 2  4  3  3  4 |
| 3  6  6  3  6 |
> rref(M);

\[
\begin{bmatrix}
1 & 2 & 0 & 3 & 0 & 2 \\
0 & 0 & 1 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

1.2 #16)
> A:=matrix([3,6,9,5,25,53,7,14,21,9,53,105,-4,-8,-12,5,-10,11]);
A:=
| 3  6  9  5  25  53 |
| 7  14  21  9  53  105 |
| -4 -8 -12  5 -10  11 |
> rref(A);

\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 5 & 6 \\
0 & 0 & 0 & 1 & 2 & 7 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

1.2 #17)
> A:=matrix([2,4,3,5,6,37,4,8,7,5,2,74,-2,-4,3,4,-5,20,1,2,2,-1,2,25,5,-10,4,6,4,241]);
A:=
| 2  4  3  5  6  37 |
| 4  8  7  5  2  74 |
| -2 -4  3  4 -5  20 |
| 1  2  2 -1  2  26 |
| 5 -10  4  6  4  24 |
> rref(A);

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -8221/4340 \\
0 & 1 & 0 & 0 & 0 & 8591/8680 \\
0 & 0 & 1 & 0 & 0 & 4695/434 \\
0 & 0 & 0 & 1 & 0 & -459/434 \\
0 & 0 & 0 & 0 & 1 & 699/434
\end{bmatrix}
\]
```