Math 2270 8/27/05

Linear system \((m \times n), \ m \ by \ n\)

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \] (*1)
\[ \vdots \]
\[ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \]

- Usually \(m = n\) (as many equations as unknowns)

\(a_{ij}\) coefficients, known

\(b_i\) right hand sides, known

\(x_i\) unknowns, to be determined.

A lot more detail throughout the semester, but (*) can be written as

\[ A\mathbf{x} = \mathbf{b} \]

where \(A\) is an \(m \times n\) matrix

containing the \(a_{ij}\), \(\mathbf{x}\) is a vector

containing the \(x_j\), and \(\mathbf{b}\) is a

vector containing the right hand side.


You don't need to understand everything.
Example 1. Polynomial interpolation

Let function $f$ be defined on $[a,b]$ and points $x_0, x_1, \ldots, x_k$ in $[a,b]$. Find a polynomial $p(x) = \sum_{j=0}^{k} a_j x^j$ such that $p(x_i) = f(x_i)$ for $i = 0, 1, \ldots, k$.

\[
\begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^k \\
1 & x_1 & x_1^2 & \cdots & x_1^k \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
1 & x_k & x_k^2 & \cdots & x_k^k
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_k
\end{bmatrix}
=
\begin{bmatrix}
f(x_0) \\
f(x_1) \\
f(x_k)
\end{bmatrix}
\]

In the $(\#)$ notation

\[
\begin{align*}
\bar{n} &= \\
\bar{a}_{\bar{m}\bar{j}} &= \\
\bar{b}_i &= \\
\bar{x}_i &=
\end{align*}
\]

It's commonplace to have to switch notation...
One can show in various ways that this problem has a unique solution if all the \( x_i \) are distinct.

We'll learn about the necessary tools in this semester.

Example 2. Sometimes a linear system does not look linear. Suppose for example we want to interpolate by a rational function

\[
R(x) = \frac{\sum_{j=0}^{M} a_j x^j}{\sum_{j=0}^{N} b_j x^j} = \frac{p(x)}{q(x)}
\]

we have parameters

So \( n = \)

\[
R(x) = f(x; i) \text{ does not look like a linear equation, but the system becomes linear after multiplying with } q(x)
\]

\[
\sum_{j=0}^{M} a_j x^i - f(x; i) \sum_{j=0}^{N} b_j x^j = 0
\]
In this case,

\[ n = \]

\[ a_{ij} = \]

\[ b_i = \]

\[ x_i = \]

Can anybody pinpoint a major difference between Examples 1 and 2?

Linear systems can become very large. Example: Numerical Solution of Boundary Value problems:

\[ y'' = f(x) \quad y(0) = A \quad y(b) = B \]

\[ h = \frac{1}{n+1} \quad x_k = kh \quad y_k \approx y(x_k) \]

The second derivative can be approximated:

\[ y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \]
- get the linear system

\[
\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = f(x_k)
\]

\[y_0 = A \quad y_n = B\]

- This system has special properties.
  (It's symmetric, tridiagonal, and negative definite)

- There's a corresponding partial differential equation

\[u_{xx} + u_{yy} = f \quad 0 \leq x, y \leq 1\]

\[x_i = ih \quad y_j = jh\]

\[U_{ij} \approx u(x_i, y_j)\]

\[-4U_{ij} + U_{i+1,j} + U_{i-1,j} + U_{ij+1} + U_{ij-1} \quad \quad h^2\]

- Ex. 4

\[- h = \frac{1}{1000}\]

If \( h = \frac{1}{1000} \) this is a linear system of 1,000,000 equations in 1 million unknowns.

- Matrix again is very special!
Example 5.

\[ F(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \end{bmatrix} = 0 \]

Newton's method replaces this with an infinite sequence of linear systems (that hopefully converges quickly).

The coefficient matrix of these linear systems is the Jacobian, i.e., the matrix of partial derivatives \( \frac{\partial f_i}{\partial x_j} \).

Linear systems arise in physical problems, Example 6. Kirchhoff's laws: (about electric circuits)

- The sum of currents flowing into a node is zero
- The sum of voltage gains and drops in a closed loop is zero
Example 6. The Transportation problem

Optimize distribution of a commodity (like coal, say)

- $M$ sources with capacities $S_i; \ i = 1, \ldots, M$
- $N$ users with demand $U_j; \ j = 1, \ldots, N$

Suppose total production equals total demand, for simplicity.

- $x_{ij}$ amount moved from source $i$ to user $j$

$$\sum_{j=1}^{N} x_{ij} = S_i; \ i = 1, \ldots, M$$
$$\sum_{i=1}^{M} x_{ij} = U_j; \ j = 1, \ldots, N$$

This system is not square and has infinitely many solutions. Pick the one that minimizes

$$\sum_{i,j} c_{ij} x_{ij}$$

where $c_{ij}$ is the cost of moving one unit of the commodity from source $i$ to user $j$. 
Example 7  Population Dynamics

Applies to all kinds of populations, for simplicity, let's think in terms of people.

\( p_n(t) \): # of people at time \( t \) that are \( n \) years old.

People are born at certain rate:

\[
p_0(t+1) = \sum_{n=0}^{200} A_n p_n(t)
\]

and die at certain rate:

\[
p_{n+1}(t+1) = \beta_n p_n(t) \quad 0 \leq \beta_n \leq 1
\]

Can be written as a matrix equation:

\[
p(t+1) = A p(t)
\]

Stable population:

\[
p(t) = p(t+1) = p
\]

\[
p = A p
\]

This is an eigenvalue problem.