Math 2270 8/27/05

- Linear system ($m \times n$, m by n)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \qquad (*)$$

- usually $m = n$ (as many equations as unknowns)

- a_{ij} coefficients, known
  b_i right hand sides, known
  x_i unknowns, to be determined.

- A lot more detail throughout the semester, but (*) can be written as

$$Ax = b$$

where A is an $m \times n$ matrix containing the a_{ij}, x_i is a vector containing the x_j, and b is a vector containing the right hand sides.

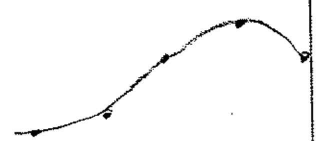
- Today: Examples. Exercise: fill in details. you don't need to understand everything, just get the flavor.

- Example 1. Polynomial interpolation

  function f defined on $[a,b]$

  points $x_0, x_1, \ldots, x_k$ in $[a,b]$

- Find a polynomial $p(x) = \sum_{j=0}^{k} \alpha_j x^j$
such that $p(x_i) = f(x_i)$
$i = 0, \ldots, k$



$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^k \\ 1 & x_1 & x_1^2 & \cdots & x_1^k \\ \vdots & & & & \\ 1 & x_k & x_k^2 & \cdots & x_k^k \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \\ \\ \alpha_k \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \\ \\ f(x_k) \end{bmatrix}$$

- In the (*) notation

  $n =$

  $a_{ij} =$

  $b_i =$

  $x_i =$

- It's commonplace to have to switch notation

- One can show in various ways that this problem has a unique solution if all the x_i are distinct.

- We'll learn about the necessary tools in this semester.

- Example 2. Sometimes a linear system does not look linear. Suppose for example we want to interpolate by a rational function

$$R(x) = \frac{\sum_{j=0}^{M} \alpha_j x^j}{\sum_{j=0}^{N} \beta_j x^j} = \frac{p(x)}{q(x)}$$

- we have parameters

- So $n =$

- $R(x_i) = f(x_i)$ does not look like a linear equation, but the system becomes linear after multiplying with $q(x)$

$$\sum_{j=0}^{M} \alpha_j x_i^j - f(x_i) \sum_{j=0}^{N} \beta_j x_i^j = 0$$

- In this case,

$n =$

$a_{ij} =$

$b_i =$

$x_i =$

- Can anybody pinpoint a major difference between Examples 1 and 2?

- Linear systems can become very large. Example: Numerical Solution

Ex. 3 of Boundary Value problems.

$$y'' = f(x) \qquad y(0) = A \qquad y(b) = B$$

$$h = \frac{1}{n+1} \qquad x_k = kh \qquad y_k \approx y(x_k)$$

- The second derivative can be approximated:

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

- get the linear system

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = f(x_k)$$

$$y_0 = A \qquad y_n = B$$

- This system has special properties. (It's symmetric, ~~tridiagonal~~ and negative definite)

- There's a corresponding partial differential equation

$$u_{xx} + u_{yy} = f \qquad 0 \le x, y \le 1$$

$$x_i = ih \quad y_j = jh$$

$$U_{ij} \approx u(x_i, y_j)$$

Ex. 4

$$\frac{-4U_{ij} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}}{h^2} = f(x_i, y_j)$$

if $h = \frac{1}{1000}$ this is a linear system of 1,000,000 equations in 1 million unknowns.

- matrix again is very special

- Newton's method Example 5.

- Nonlinear system

$$F(x) = \begin{bmatrix} f_1(x_1 \dots x_n) \\ \\ f_n(x_1 \dots x_n) \end{bmatrix} = 0$$

- Newton's method replaces this with an infinite sequence of linear systems (that hopefully converges quickly)

- The coefficient matrix of those linear systems is the Jacobian, i.e., the matrix of partial derivatives $\frac{\partial f_i}{\partial x_j}$

- Linear systems arise in physical problems. Example 6. Kirchoff's laws: (about electric circuits)
  - The sum of currents flowing into a node is zero
  - The sum of voltage gains and drops in a closed loop is zero

Example 6. <u>The</u> Transportation problem

- Optimize distribution of a commodity (like coal, say)

M sources with capacities S_i $i=1,\dots,M$
N users with demand U_j $j=1,\dots,N$

- Suppose total production equals total demand, for simplicity.

- x_{ij} amount moved from source i to user j

$$\sum_{j=1}^{N} x_{ij} = S_i \qquad i=1,\dots,M$$
$$\sum_{i=1}^{M} x_{ij} = U_j \qquad j=1,\dots,N$$

- This system is not square and has infinitely many solutions.
Pick the one that minimizes

$$\sum_{i,j} c_{ij} x_{ij}$$

where c_{ij} is the cost of moving one unit of the commodity from source i to user j.

- Example 7 Population Dynamics
- Applies to all kinds of populations, for simplicity, let's think in terms of people.

$$P_n(t): \text{\# of people at time } t \text{ that are } n \text{ years old.}$$

- People are born at certain rates:

$$P_0(t+1) = \sum_{n=0}^{200 \text{ (say)}} \alpha_n P_n(t)$$

and die at certain rates:

$$P_{n+1}(t+1) = \beta_n P_n(t) \qquad 0 \le \beta_n \le 1$$

- Can be written as a matrix equation

$$p(t+1) = A\, p(t)$$

- stable population $\quad p(t) = p(t+1) = p$

$$p = A p$$

- this is an eigenvalue problem.