**Finding bases for image(f) and ker(f)**

We work with the vectors from problem 24 in section 3.3; page 131.
```
> restart:
> with(linalg):
 Warning, the protected names norm and trace have been redefined and unprotected
> At:=matrix(5,5,[1,2,3,2,1,3,6,9,6,3,3,2,4,1,2,8,4,9,1,5,
   0,4,5,5,1]);
   #this is the "transpose" of the matrix
   #I want, i.e. the columns and rows have been reversed.
```

$$At := \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 3 & 2 & 4 & 1 & 2 \\ 8 & 4 & 9 & 1 & 5 \\ 0 & 4 & 5 & 5 & 1 \end{bmatrix}$$

```
> A:=transpose(At); #this is the matrix I want.
   #the image of f(x)=Ax is the subspace of #24 page 131
```

$$A := \begin{bmatrix} 1 & 3 & 3 & 8 & 0 \\ 2 & 6 & 2 & 4 & 4 \\ 3 & 9 & 4 & 9 & 5 \\ 2 & 6 & 1 & 1 & 5 \\ 1 & 3 & 2 & 5 & 1 \end{bmatrix}$$

If I was doing 24, I am looking for a basis of subspace image(f), where f(x)=Ax. In 24 I want my basis to be a subset of the original 5 column vectors. These 5 vectors in the "column space" may be dependent, in which case I would like to throw away redundant vectors until I am left with a basis. I can discover dependencies from the kernel( f), which I get from rref(A):
```
> rref(A);
```

$$\begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Backsolve to find a basis for kef(f): (Think of A and rref(A) augmented by a zero column for the right hand side.) Check that you have a basis for ker(f), because this is something you need to know how to do in lots of applications.

How does your work compare to Maple:
```
> nullspace(A); #basis for the kernel of f,
    #also called the nullspace of A
```
$$\{[-3, 0, 1, 0, 1], [-3, 1, 0, 0, 0], [-8, 0, 0, 1, 3]\}$$

Actually, since column dependencies are really "the same" as solutions to Ax=0, they are unchanged as you do elementary row operations. Therefore since in the reduced matrix I have the the second column equal to 3 times the first, the same fact is true in the original matrix! Since in the reduced matrix the 4th column is 3 times the third minus the first, the same is true in the original matrix! Similarly, the 5th column is dependent on the first and third columns. Thus an acceptable basis for the span of my 5 column vectors is

```
> imbasis:={col(A,1),col(A,3)};
```
$$imbasis := \{[1, 2, 3, 2, 1], [3, 2, 4, 1, 2]\}$$

A nicer basis would have vectors with more zero entries. We can obtain this by doing elementary column operations to A, since these do not change the span of the columns! This is the same as computing
```
> transpose(rref(At));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \dfrac{1}{2} & \dfrac{5}{4} & 0 & 0 & 0 \\ \dfrac{-1}{2} & \dfrac{5}{4} & 0 & 0 & 0 \\ \dfrac{1}{2} & \dfrac{1}{4} & 0 & 0 & 0 \end{bmatrix}$$

This is how maple does it:

```
> colspace(A); #basis for column space of A,
   #i.e. for the image of f
```

$$\{\left[1, 0, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right], \left[0, 1, \frac{5}{4}, \frac{5}{4}, \frac{1}{4}\right]\}$$

It's always easy to tell that the basis you get this way is actually a basis; the independence follows easily, and for example it is also easy to express your basis obtained by deletion in terms of this col op derived basis.  Do it.