Name...
I.D. number...

Math 2270-2
Sample Final Exam SOLUTIONS

December 2001

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and **justify your conclusions.** There are 200 points possible, and the point values for each problem are indicated in the right-hand margin. Of course, this exam counts for 30% of your final grade even though it is scaled to 200 points. Good Luck!

```
> restart:with(linalg):
```

1) Let

$$A := \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Then L(**x**)=A**x** is a matrix map from **R**^3 to **R**^2.

1a) Find the four fundamental subspaces associated to the matrix A.

(20 points)

$$A := \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

```
> rref(A);
```

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

So, the row space is the span of {[1,,0,1],0,1,-1]}. The column space is the span of the first two columns of A, and this is all of R^2. The nullspace can be obtained by backsolving the homogeneous system, using rref(A). We see that x3=t, x2=t, x1=-t, so x=t[-1,1,1], so a nullspace basis is {[-1,1,1]} . Since the column space is all or R^2, its orthogonal complement is the zero vector.

1b) State and verify the theorem which relates rank and nullity of A, in this particular case

(5 points)

rank + nullity = #cols(A). In this case 2+1=3.

1c) Find an orthonormal basis for R^3 in which the first two vectors are a basis for the rowspace of A and the last vector spans its nullspace.

(10 points)

The rowspace and nullspace of A are already orthogonal, so I just need to Gramschmidt my rowspace basis, and then normalize:

$$v1 := [1, 0, 1]$$

$$v2 := [0, 1, -1]$$

```
> w1:=evalm(v1)/norm(v1,2);
  z2:=evalm(v2-(dotprod(v2,w1)*w1));
  w2:=evalm(z2)/norm(z2,2);
```

$$w1 := \frac{1}{2}[1, 0, 1]\sqrt{2}$$

$$z2 := \left[\frac{1}{2}, 1, \frac{-1}{2}\right]$$

$$w2 := \frac{1}{3}\left[\frac{1}{2}, 1, \frac{-1}{2}\right]\sqrt{6}$$

So my basis is

$$\left[\begin{bmatrix}\frac{1}{2}\sqrt{2}\\ 0\\ \frac{1}{2}\sqrt{2}\end{bmatrix}, \begin{bmatrix}\frac{1}{6}\sqrt{6}\\ \frac{1}{3}\sqrt{6}\\ -\frac{1}{6}\sqrt{6}\end{bmatrix}, \begin{bmatrix}-\frac{1}{3}\sqrt{3}\\ \frac{1}{3}\sqrt{3}\\ \frac{1}{3}\sqrt{3}\end{bmatrix}\right]$$

2a) Exhibit the rotation matrix which rotates vectors in R^2 by an angle of α radians in the counter-clockwise direction.

(5 points)

$$rot := \alpha \to \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}$$

>

2b) Verify that the product of an α-rotation matrix with a β-rotation matrix is an $(\alpha+\beta)$-rotation matrix.

(10 points)

$$\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{bmatrix} =$$

$$\begin{bmatrix}\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta)\\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{bmatrix}$$

By trig addition angle formulas we recognize this last matrix as

$$\begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

>

3) Let T be the linear map from the complex plane to the complex plane, defined by

$$\mathrm{T}(z) = (1+i)\,z$$

We can consider the complex numbers as a real vector space of dimension 2, with basis β={1,i}. Find the matrix for T with respect to this basis, and explain what T does geometrically.

(10 points)

We check what T does to our basis:

$$\mathrm{T}(1) = 1 + i$$
$$\mathrm{T}(i) = -1 + i$$

Thus the first column of our matrix B will be the coordinates of T(1) with respect to our basis, i.e. [1,1]. Similarly, the second column is [-1,1]. So our matrix B is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

*Thus T is the composition of a rotation by Pi/4 with a dilation by sqrt(2). This is consistent with our understanding of complex multiplication, since in polar form 1+i = sqrt(2)*exp(i*Pi/4).*

4a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

$$x^T A x + B x + c = 0$$

into one without any ''cross terms''. Be precise in explaining the change of variables, and the justification for why such a change of variables exists.

(10 points)

A is symmetric, so we can diagonalize it with an orthogonal matrix S , ie transpose(S)AS=D. We then let

$$x = S y$$

and the quadratic form transforms to

$$y^T \mathrm{D} y + B S y + c = 0$$

4b) Apply the procedure from part (4a) to put the conic section

$$6 x^2 + 9 y^2 - 4 x y + 4\sqrt{5}\, x - 18\sqrt{5}\, y = 5$$

into standard form. Along the way, identify the conic section.

(20 points)

$$A := \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$$

```
> eigenvects(A);
```

[10, 1, {[1, -2]}], [5, 1, {[2, 1]}]

Since both eigenvalues are positive, we have an ellipse. Continuing, we can take

$$S := \frac{1}{5}\sqrt{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

And, using u,v for our new variables, our transformed equation is

$$10 u^2 + 5 v^2 + 40 u - 10 v = 5$$

we may complete the square to get

$$5 (v - 1)^2 - 45 + 10 (u + 2)^2 = 5$$

$$5\,(v-1)^2 + 10\,(u+2)^2 = 50$$

$$\frac{1}{10}\,(v-1)^2 + \frac{1}{5}\,(u+2)^2 = 1$$

5) Let

$$C := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

5a) Find the inverse of C using elementary row operations.

(15 points)

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

```
> rref(%);
```

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & 4 & 3 & -2 \end{bmatrix}$$

So the inverse matrix is

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 4 & 3 & -2 \end{bmatrix}$$

5b) Find the inverse of C using the adjoint formula.

(15 points)

$$\mathrm{cof}(C) := \begin{bmatrix} 1 & 2 & -4 \\ 1 & 1 & -3 \\ -1 & -1 & 2 \end{bmatrix}$$

```
> adj(C):=transpose(cof(C));
```

$$\mathrm{adj}(C) := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

We calculate that det(C)=2-2-1=-1, so the inverse is given by

```
> detC:=-1:
  (1/detC)*adj(C);
```

$$detC := -1$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -3 & 2 \end{bmatrix}$$

6a) Define what it means for a transformation (function) T:V-->W between vector spaces to be a **linear transformation**.

(4 points)

I) T(u+v)=T(u)+T(v) for all u,v in V
II) T(cu)=cT(u) for all u in V, c in R.

6b) Define what it means for a set $\beta=\{\mathbf{v}_1,\mathbf{v}_2, \ldots \mathbf{v}_n\}$ to be a **basis** for a vector space V.

(3 points)

β is a basis for V if it is linearly independent and spans V.

6c) For a linear map T as in part (6a), define the **kernel** of T.

(3 points)

The kernel of T is the set of all vectors in V which satisfy T(v)=0.

6d) Let $\beta=\{\mathbf{v}_1,\mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for V T::V-->V be linear. Explain what the matrix for T with respect to β is, and what it does.

(5 points)

*The matrix for T with respect to β is the matrix B which has the property that B times the β-coordinates of a vector v in V equals the β-coordinates of T(v), for all v in V. The jth column of B is just the coordinates of T(**v**j), with respect to T. (So that's one way to compute it.)*

7) Let

$$A := \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}$$

7a) Let

$$\beta := \left[\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right]$$

(Note these are the columns of your matrix C from #5.) Let T(x)=Ax, a matrix map from R^3 to R^3. Thus A is the matrix of T with respect to the standard basis of R^3. Find the matrix of T with respect to the β basis.

(15 points)

Method I: The matrix for L with respect to β *is the triple product* $[\boldsymbol{P}_{\beta\text{-}<E}][\boldsymbol{A}][\boldsymbol{P}_{E<\text{-}\beta}]$. *We write S for the transition matrix* $[\boldsymbol{P}_{E<\text{-}\beta}]$.

$$A := \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}$$

$$S := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ -2 & -1 & 1 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Method 2: For β*={v1,v2,v3} compute {Av1,Av2,Av3} and then find coords with respect to* β*. It turns out that Av1=3v1, Av2=2v3, Av3=v2. So it is easy to write down the matrix for T exhibited above.*

In general, this leads to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 0 & 3 & 0 & -1 \\ 0 & 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 1 & 6 & 2 & 1 \end{bmatrix}$$

```
> rref(%);
```

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

From which we take the last 3 columns as our matrix.

7b) Let

$$v := 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Compute T(v) two ways: once using the matrix for T with respect to β, and once using the matrix A. Verify that your answers agree.

(10 points)

Method 1: Use the E-coordinates for v

$$\begin{bmatrix} -7 & -6 & 5 \\ 4 & 4 & -2 \\ -6 & -5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}=\begin{bmatrix} 7 \\ 0 \\ 13 \end{bmatrix}$$

Method 2: Use β-coordinates

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}=\begin{bmatrix} 7 \\ 0 \\ 13 \end{bmatrix}$$

8) True-False. Four points each (2 points for answer, two points for justification.)

(40 points)

8i) Let A be an n by n matrix. Then if Ax=Ay it follows that x=y.
FALSE: would only be true if A was non-singular
8i) If A and B are n by n matrices, then

$$(A+B)^2 = A^2 + 2\,A\,B + B^2$$

FALSE: would only be true if AB=BA

8iii) If A is any matrix with (real entries) then the product

$$B = A^T A$$

has only real eigenvalues, and is in fact similar to a diagonal matrix.
True: note that B is symmetric so the spectral theorem implies the claims.

8iv) Every set of five orthonormal vectors in **R**^5 is automatically a basis for **R**^5.
TRUE: orthonormal vectors are automatically linearly independent, and 5 independent vectors automatically span a 5-dimensional space

8v) The number of linearly independent eigenvectors of a matrix is always greater than or equal to the number of distinct eigenvalues.
TRUE: each eigenspace is at least one-dimensional, and the union of eigenspace bases is still linearly independent.

8vi) If the rows of a 4 by 6 matrix are linearly dependent then the nullspace is at least three dimensional.
TRUE: rows dependent so row rank is at most 3. Since row rank plus nullity equals 6 this means nullity is at least 3

8vii) A diagonalizable n by n matrix must always have n distinct eigenvalues.
FALSE; e.g. the identity matrix
8viii) The equation

$$2\,x^2 + 5\,x\,y + 3\,y^2 = 1$$

defines an ellipse.
FALSE. The matrix and eigenvalues of the quadratic form are

```
> A:=matrix(2,2,[2,5/2,5/2,3]);
  Evals:=eigenvals(A);
```

$$A := \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 3 \end{bmatrix}$$

$$Evals := \frac{5}{2} + \frac{1}{2}\sqrt{26},\ \frac{5}{2} - \frac{1}{2}\sqrt{26}$$

Since theevals have opposite sign, what we really have is a hyperbola!
8ix) If A and B are orthogonal matrices then so is AB.
TRUE: use the fact that the transpose of AB is B transpose time A transpose.

8x) If A is a square matrix and A^2 is singular, then so is A.
TRUE: if det(A^2)=0, then this also equals (det(A))^2 by multiplicative determinant property, so det(A)=0.